Version DC/3

EGT2
ENGINEERING TRIPOS PART IIA

Friday 29 April 2022 9.30 to 11.10

Module 3C6

VIBRATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Attachment: 3C5 Dynamics and 3C6 Vibration data sheet 2021 (7 pages).

Engineering Data Book.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 As a first approximation axial vibration of a space launch vehicle is modelled as a rod of length L and cross-sectional area A, with mass per unit length ρ and Young's modulus E. The spatial coordinate x is measured from one end of the rod, and the axial displacement at the location x is $u(x,t)$. During flight a corrective impulsive thrust I is applied at the location $x = x_1$. Just after the impulse (at $t = 0$ say) the resulting conditions are

$$u(x,0) = 0, \qquad \dot{u}(x,0) = (I / \rho)\delta(x - x_1).$$

The rod has free boundary conditions at each end.

(a) Assume a solution for the resulting motion of the rod in the form

$$u(x,t) = f(x - ct) + g(x + ct),$$

where c is the compressive wave speed. Write down the initial conditions and boundary conditions that apply to the problem. By focusing on the properties of $f'(x)$, rather than $f(x)$, find an expression for $f'(x)$ over the range $0 \leq x \leq L$ and show that beyond this range the function is symmetric and has period $2L$. Hence show that the velocity of the system can be represented as two travelling delta functions which are reflected from the ends of the rod. [35%]

(b) Plot the velocity at $x = x_1$ as a function of time, taking the case $x_1 = L/2$. Integrate this result to produce a plot of the time history of the displacement at this point. By considering the change in displacement over a time period $2L/c$, find the average velocity of the system. Compare this result to that which would be obtained were the impulse applied to a rigid rod. [30%]

(c) Write down an expression for the *velocity* response of the system at location x in terms of the system mode shapes and natural frequencies (you do not need to derive these quantities, simply use the symbols ϕ_n and ω_n). Derive an expression for the time history of the kinetic energy of the rod and simplify your result by noting the orthogonality of the mode shapes. [20%]

(d) Show that your answer to part (c) leads to the conclusion that the kinetic energy is infinite at $t = 0$. Clearly this result is non-physical; explain the extent to which the results obtained in parts (a)-(c) will approximate the behaviour of a real rod. [15%]

2 A string of length L, mass per unit length m, and tension T, is subjected to a harmonic force $F\cos\omega t$ applied at the point $x = x_1$ as shown in Fig. 1.

(a) Show that the response of the string has the form

$$u(x,t) = \begin{cases} A_1 \sin kx \cos\omega t & 0 \le x \le x_1 \\ A_2 \sin[k(x-L)]\cos\omega t & x_1 \le x \le L \end{cases}$$

where A_1 and A_2 are constants. Derive an expression for k in terms of ω and the properties of the string. [10%]

(b) Calculate the response of the system at $x = x_1$ and deduce the natural frequencies of the system. Show that your result for the response tends to the correct static solution for $\omega \to 0$. [30%]

(c) Using results for $u(x_1,t)/F$ and $u(x_2,t)/F$, with $x_2 > x_1$, find the response at x_2 arising from a prescribed *displacement* $A\cos\omega t$ at x_1. Find the frequencies at which the response has peaks, and give a physical interpretation of your result. [20%]

(d) The mode shapes and natural frequencies of the string have the form

$$\phi_n(x) = C\sin\left(\frac{n\pi x}{L}\right), \qquad \omega_n = \left(\frac{n\pi}{L}\right)\sqrt{\frac{T}{m}}.$$

Find the constant C which ensures that the mode shapes are normalised. By using the modal response formula given on the data sheet, express the response quantity considered in part (b) in terms of a modal sum. [20%]

(e) The analytical and the modal solutions to part (b) are mathematically identical, although this is not obvious. Show that the two approaches give the same answer for the case where $\omega = \omega_5 + \delta$ where δ is very small. [20%]

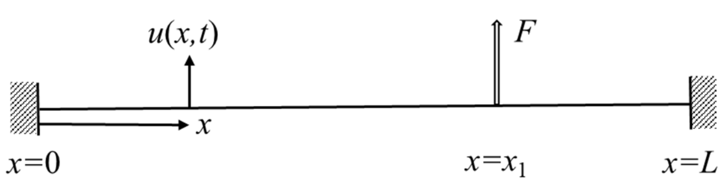

Fig. 1

3 Figure 2 shows a 'pitch-plane' model of a lorry, subject to small vibrations. The wheels are represented by two masses m, which are constrained to move vertically with displacements x_1 and x_2. The body is symmetric, with mass $6m$ and pitch moment of inertia $I = 2ma^2$. It can move vertically with displacement y and pitch with angle θ. The masses are connected by four linear springs, each of stiffness k, representing the tyres and suspension springs. The distance between the two axles is $2a$.

(a) Assuming small motions, write down expressions for the kinetic and potential energies T and V in terms of the coordinates x_1, x_2, y and θ. Hence write down the mass matrix and show that the stiffness matrix can be written:

$$[K] = k\begin{bmatrix} 2 & 0 & -1 & a \\ 0 & 2 & -1 & -a \\ -1 & -1 & 2 & 0 \\ a & -a & 0 & 2a^2 \end{bmatrix}$$

where the vector of generalised coordinates is $\begin{bmatrix} x_1 & x_2 & y & \theta \end{bmatrix}^T$. [20%]

(b) The system has two natural modes with eigenvectors of the form $\begin{bmatrix} 1 & 1 & \alpha & 0 \end{bmatrix}^T$ and two natural modes with eigenvectors of the form $\begin{bmatrix} 1 & -1 & 0 & \beta \end{bmatrix}^T$. By substituting these mode shapes into the eigenvalue calculation, or otherwise, determine the values of α and β, and determine the four natural frequencies. Sketch the four corresponding mode shapes. [50%]

(c) Sketch log amplitude plots for the transfer functions describing the displacements of coordinates x_1 and y when a sinusoidal force f is applied to the left wheel as shown in the figure. [30%]

(cont.

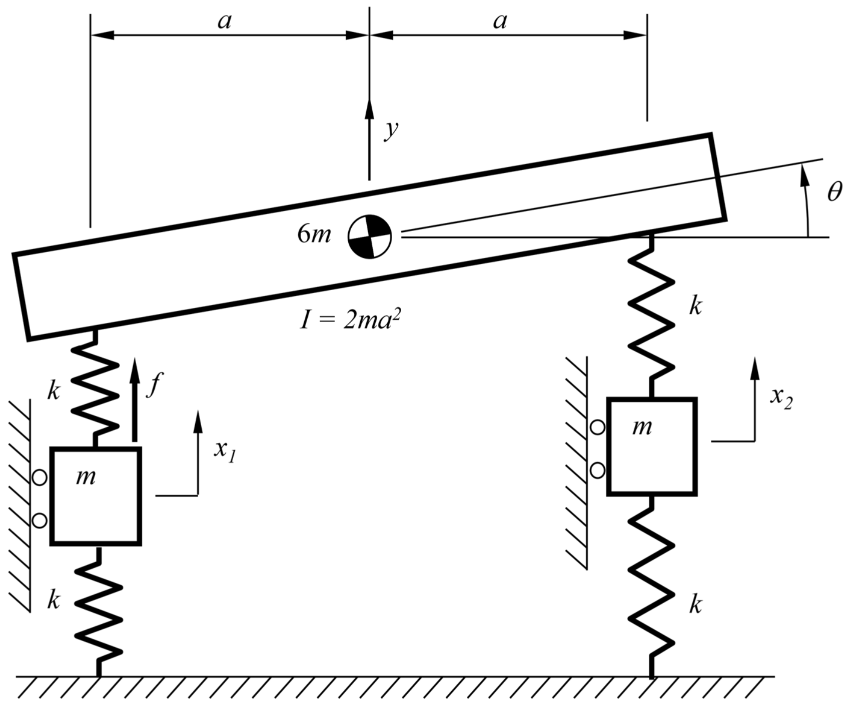

Fig. 2

4 Three particles of mass *m, 2m, m* are attached to a tightly-stretched, light wire as shown in Fig. 3. The tensile force P in the wire is large, so that it does not change appreciably for small lateral displacements of the particles. The length of each segment of the wire is L as shown.

(a) Show that the potential energy of the system is given by:

$$V = \frac{P}{L}\left[y_1^2 + y_2^2 + y_3^2 - y_1 y_2 - y_2 y_3\right],$$

and write down the stiffness and mass matrices. [25%]

(b) Sketch the natural mode shapes. Hence, or otherwise, calculate the natural frequencies. [40%]

(c) If the middle mass is decreased by 20%, use Rayleigh's quotient to estimate the percentage change in the lowest natural frequency. Explain how you could use Rayleigh's quotient to obtain the exact answer. [35%]

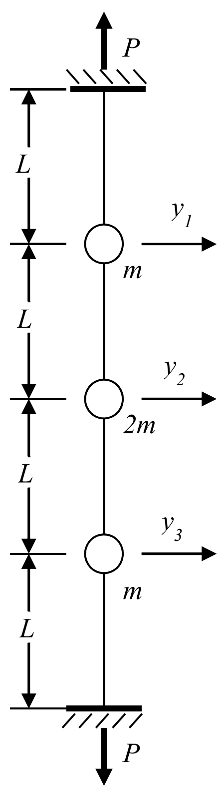

Fig. 3

END OF PAPER

Answers

1. (a) $f'(x) = -g'(x) = -(I/2\rho c)\delta(x - x_1)$, $f'(-ct) = f'(ct)$, $f'(ct - L) = f'(ct + L)$

   $\dot{u}(x,t) = (I/2\rho)\{\delta(x - x_1 - ct) + \delta(x - x_1 + ct)\}$

   (b) Average velocity is $I/\rho L$, the same result as a rigid rod.

   (c) $\dot{u}(x,t) = I\sum_n \phi_n(x)\phi_n(x_1)\cos\omega_n t$, $T = (I^2/2)\sum_n \phi_n^2(x_1)\cos^2\omega_n t$.

   (d) From the above summation T is infinite at t=0. The delta function impulse is an unrealistic simplification from this point of view.

2. (a) $k = \omega\sqrt{m/T}$.

   (b) $u(x,t) = -(F/Tk)\cos\omega t \sin kx_1 \sin k(x - L)/\sin kL$ for $x > x_1$.

   (c) $u = A\cos\omega t \sin k(x_2 - L)/\sin k(x_1 - L)$, $\omega_n = n\pi\sqrt{T/m}/(L - x_1)$.

   (d) $C = \sqrt{2/mL}$, $u(x_1,t) = \left(\dfrac{2F\cos\omega t}{mL}\right)\sum_n \dfrac{\sin^2(n\pi x_1/L)}{\omega_n^2 - \omega^2}$

   (e) Both results tend to $-(F/5\pi\delta)\sin^2(5\pi x_1/L)/\sqrt{mT}$.