3C6 Cribs 2022

1. a) $$u(x,t) = f(x-ct) + g(x+ct)$$

Initial conditions : $u(x,0) = 0$ — (1)

$\dot{u}(x,0) = (I/\rho)\,\delta(x-x_1)$ — (2)

Boundary conditions : $u'(0,t) = 0$ — (3)

$u'(L,t) = 0$ — (4)

$\Big\}$ Ends are free, so that $EAu' = 0$

(1) $\Rightarrow$ $f(x) + g(x) = 0 \Rightarrow g(x) = -f(x)$

(2) $\Rightarrow$ $-cf'(x) + cg'(x) = (I/\rho)\,\delta(x-x_1)$

$$\Rightarrow \quad f'(x) = -(I/2\rho c)\,\delta(x-x_1)$$
$$g'(x) = (I/2\rho c)\,\delta(x-x_1)$$

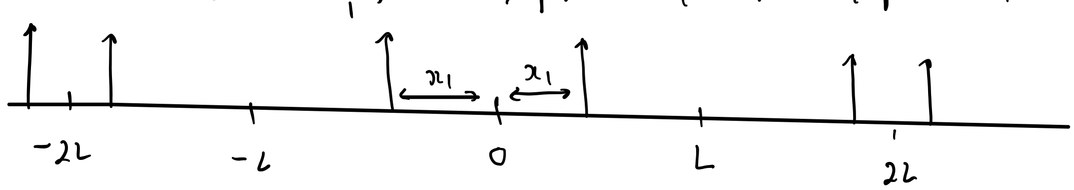
Now (3) $\Rightarrow$ $u'(0,t) = f'(-ct) + g'(ct) = 0$

$\Rightarrow$ $f'(-ct) = f'(ct) \Rightarrow f'$ is <u>symmetric</u>

And (4) $\Rightarrow$ $u'(L,t) = f'(L-ct) + g'(L+ct) = 0$

$f'(ct-L) = f'(ct+L) \Rightarrow f'$ <u>has period $2L$</u>

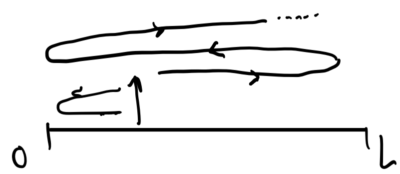
$$\Rightarrow \quad \dot{u}(x,t) = (I/2\rho)\,\delta(x-x_1-ct) + (I/2\rho)\,\delta(x-x_1+ct)$$



$\longrightarrow$ F' moves with velocity c

$\longleftarrow$ g' moves with velocity $-c$

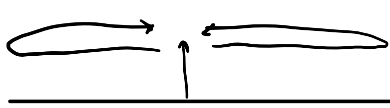
Equivalent to

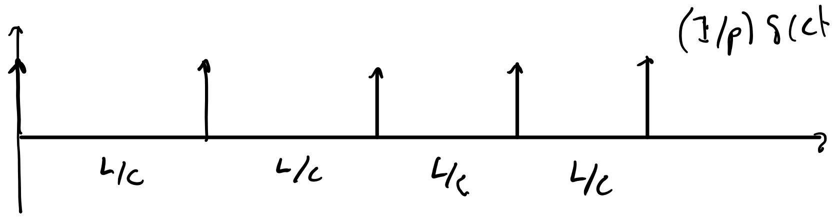


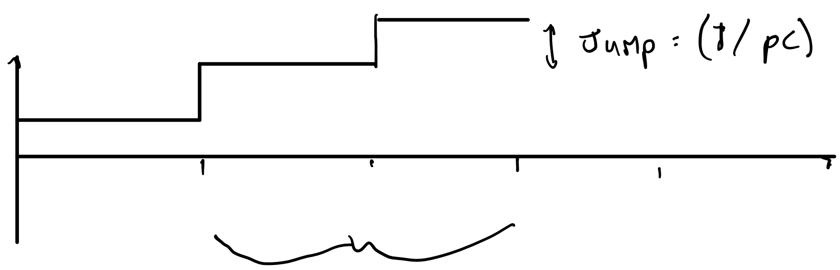
Travelling δ function in each direction

[35%]

b)







Jump $= (I/\rho c)$ since $\int \delta(ct - nL)\,dt = 1/c$

Change over $2L/c$ is $2I/\rho c$

$$\Rightarrow \text{ Average velocity } = \frac{2I}{\rho c} \times \frac{c}{2L} = I/\rho L = \left(\frac{\text{Impulse}}{\text{Mass}}\right)$$

<u>Same result</u> as rigid rod

[30%]

c)

$$u = \sum_n \frac{\phi_n(x)\phi_n(x_1)}{\omega_n} \sin\omega_n t$$

$$\dot{u} = \sum_n \phi_n(x)\phi_n(x_1) \cos\omega_n t$$

$$T = \tfrac{1}{2}\rho \int_0^L \dot{u}^2\,dx = \tfrac{1}{2}\rho \int_0^L \sum_n \sum_m \phi_n(x)\phi_n(x_1)\phi_m(x)\phi_m(x_1) \cos\omega_n t \cos\omega_m t \, dx$$

But $\rho\int_0^L \phi_n(x)\phi_m(x)\,dx = \delta_{nm}$

$$\Rightarrow \underline{T = \tfrac{1}{2}\sum_n \phi_n^2(x_1)\cos^2\omega_n t} \qquad [20\%]$$

d) $T(0) = \tfrac{1}{2}\sum_n \phi_n^2(x_1) \longrightarrow \infty$ for infinite summation.

Approximations made : (i) The impulse is ideal, and therefore contains energy at <u>all</u> frequencies

(ii) The "modes" exist in a classical sense regardless of the wavenumber. In fact the simple theory no longer applies at very short wavelengths, and quantum mechanics comes in as $n \to \infty$

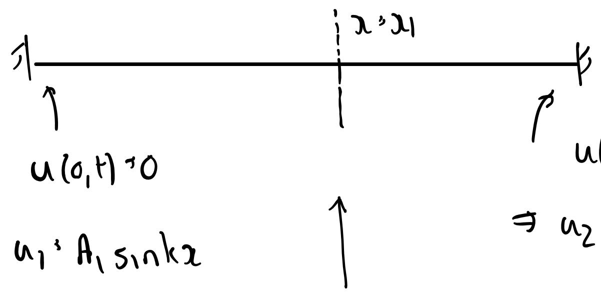
In reality the delta function will be spread in time (and space), and the modal sum will be truncated. Likewise the rod will not see step function changes in displacement, but a smoother version [15%]

— // —

2. a) $$Tu'' - m\ddot{u} = 0$$

Assume $u = e^{-ikx} e^{i\omega t}$ $\Rightarrow$ $Tk^2 = m\omega^2$ $\Rightarrow$ $\underline{k = \sqrt{\frac{m}{T}}\,\omega}$

General solution (omitting $\cos\omega t$) : $u = A\cos kx + B\sin kx$



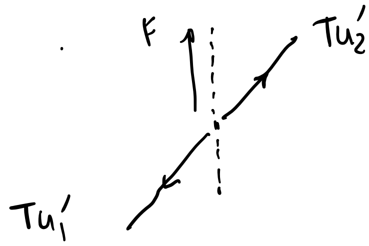
$u(0,t) = 0$ $\quad$ $u(L,t) = 0$

$\Rightarrow u_1 = A_1 \sin kx$ $\quad$ $\Rightarrow u_2 = A_2 \sin k(x-L)$

Boundary conditions to be matched at this point

[10%]

b) Boundary conditions at $x = x_1$:



$u_1 = u_2$

$F = T(u_1' - u_2')$

$$\Rightarrow \quad A_1 \sin kx_1 = A_2 \sin k(x_1 - L)$$

$$\frac{F}{Tk} = A_1 \cos kx_1 - A_2 \cos k(x_1 - L)$$

$$\Rightarrow \quad \frac{F}{Tk} = A_2 \left\{ -\cos k(x_1 - L) + \frac{\sin k(x_1-L)\cos kx_1}{\sin kx_1} \right\}$$

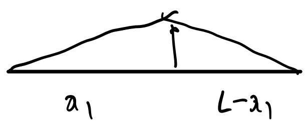
$$\Rightarrow \quad A_2 = -\left(\frac{F}{Tk}\right)\frac{\sin kx_1}{\sin kL} \quad \leftarrow \text{denominator from trig}$$

$$\Rightarrow \quad \underline{u(x_1,t) = -\left(\frac{F}{Tk}\right)\frac{\sin kx_1 \sin k(x_1-L)}{\sin kL}}$$

Infinite at $k = \frac{n\pi}{L} \Rightarrow \underline{\omega_n = \left(\frac{n\pi}{L}\right)\sqrt{\frac{T}{\rho}}}$

For $k \to 0$ $\quad u(x_1,t) \to -\left(\frac{F}{Tk}\right)\frac{k^2 x_1(x_1-L)}{kL} = -\frac{F}{T}\left[\frac{x_1(x_1-L)}{L}\right]$

Static solution



$$F = T\left[\frac{u}{x_1} + \frac{u}{L-x_1}\right]$$

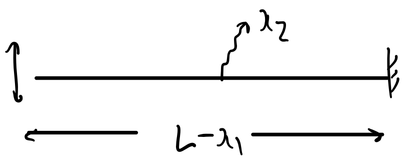
$u = \underline{\text{same result}}$ ✓ [30%]

c) $\quad u(x_1,t)/F = \left(-\frac{F}{Tk}\right)\frac{\sin kx_1 \sin k(x_1-L)}{\sin kL}$

$$u(x_2,t)/F = \left(-\frac{F}{Tk}\right)\frac{\sin kx_1 \sin k(x_2-L)}{\sin kL}$$

$$\frac{u(x_2,t)}{u(x_1,t)} = \frac{\sin k(x_2-L)}{\sin k(x_1-L)} \rightarrow \text{peaks at } k = \frac{n\pi}{L-x_1}$$

motion



resonances of string of length $L-x_1$, i.e. point of prescribed motion fixed.

[20%]

d) From part (a) $\varphi_n = \sin kx$, BC's $\Rightarrow k_n = \frac{n\pi}{L}$

$$\Rightarrow \omega_n = \left(\frac{n\pi}{L}\right)\sqrt{\frac{T}{\rho}}$$

Now $m\int_0^L \varphi_n^2(x)\,dx = 1 \Rightarrow mC^2\int_0^L \sin^2\frac{n\pi x}{L}\,dx = 1$

$$\Rightarrow mC^2\left(\tfrac{1}{2}L\right) = 1 \qquad \underline{C = \sqrt{2/mL}}$$

From the data sheet :-

$$u(x_1,t) = \sum_n \frac{\varphi_n^2(x_1)F}{\omega_n^2-\omega^2} = \underline{\left(\frac{2}{\rho L}\right)\sum_n \frac{\sin^2(n\pi x_1/L)F}{\omega_n^2-\omega^2}} \qquad [20\%]$$

e) For $\omega = \omega_5+\delta \Rightarrow \omega_5^2-\omega^2 = \omega_5^2-(\omega_5^2+2\omega_5\delta+\delta^2) \simeq -2\omega_5\delta$

Modal response $\quad u(x_1,t) \simeq \left(\frac{2}{mL}\right)\frac{\sin^2(5\pi x_1/L)}{-2\omega_5\delta}F$

$$u(x_1,t) \simeq \left(\frac{F}{L}\right)\left(\frac{L}{5\pi}\right)\frac{1}{\sqrt{mT}}\sin^2(5\pi x_1/L)\left(\frac{-1}{\delta}\right) \quad -(A)$$

Analytical response $\quad u(x_1,t) = -\frac{F}{Tk}\frac{\sin kx_1\sin k(x_1-L)}{\sin kL}$

$$k = \frac{5n\pi}{L} + \delta\sqrt{\frac{m}{T}} \qquad \sin kL \simeq -\delta\sqrt{\frac{m}{T}}L$$

$$\sin k(x_1-L) = -\sin k_5x_1$$

(A) and (B) same result

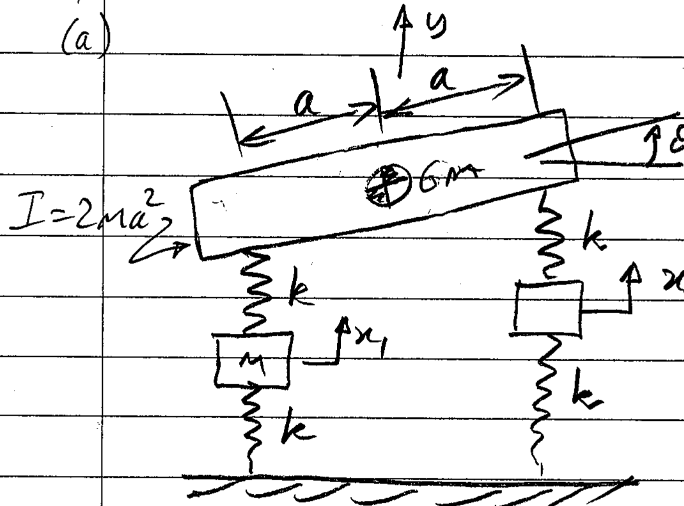
$$u(x_1,t) = \frac{F}{Tk_5}\sqrt{\frac{T}{m}}\sin^2 k_5x_1\left(\frac{-1}{\delta L}\right)$$

$$= \left(\frac{F}{L}\right)\left(\frac{L}{5\pi}\right)\frac{1}{\sqrt{mT}}\sin^2 k_5x\left(\frac{-1}{\delta}\right) \quad -(B)$$

[20%]

3 ~~4~~.

(a)



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} 6m \dot{y}^2 + \frac{1}{2} 2ma^2 \dot{\theta}^2$$

$$[M] = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 6m & 0 \\ 0 & 0 & 0 & 2ma^2 \end{bmatrix} \begin{matrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \\ \dot{\theta} \end{matrix}$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k (y - a\theta - x_1)^2 + \frac{1}{2} k (y + a\theta - x_2)^2$$

$$= \frac{1}{2} k \left(2x_1^2 + 2x_2^2 + 2y^2 + 2a^2\theta^2 - 2x_1 y - 2x_2 y + 2a x_1 \theta - 2a x_2 \theta \right)$$

Hence, by inspection of the quadratic form

$$V = \frac{1}{2} k [x_1 \ x_2 \ y \ \theta] \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & y & \theta \end{array} \\ \begin{bmatrix} 2 & 0 & -1 & a \\ 0 & 2 & -1 & -a \\ -1 & -1 & 2 & 0 \\ a & -a & 0 & 2a^2 \end{bmatrix} \\ [K] \end{array} \begin{Bmatrix} x_1 \\ x_2 \\ y \\ \theta \end{Bmatrix}$$

(or do this by Lagrange)

(b) for the natural modes $([K] - \omega^2 [M]) \underset{\sim}{u} = \underset{\sim}{0}$

for $\underset{\sim}{u} = [1 \ \ 1 \ \ \alpha \ \ 0]^T$ this is:

$$\begin{bmatrix} 2k - \omega^2 m & 0 & -k & ak \\ 0 & 2k - \omega^2 m & -k & -ak \\ -k & -k & 2k - 6\omega^2 m & 0 \\ ak & -ak & 0 & 2a^2(k - \omega^2 m) \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ \alpha \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{—} (1)$$

Unknowns α & ω^2

Row 1 of (1): $2k - \omega^2 m - k\alpha = 0 \Rightarrow (2 - \alpha) k = \omega^2 m$ — (2)

Row 3 of (1): $-2k + (2k - 6\omega^2 m)\alpha = 0 \Rightarrow (\alpha - 1) k = 3\omega^2 m \alpha$ — (3)

$\frac{(2)}{(3)} \Rightarrow \frac{2 - \alpha}{\alpha - 1} = \frac{1}{3\alpha} \Rightarrow 3\alpha^2 - 5\alpha - 1 = 0$

$\Rightarrow \alpha = \frac{5 \pm \sqrt{37}}{6}$ ✓ $\quad (2) \Rightarrow \omega^2_{1,3} = \begin{cases} 2.18 \ k/m \\ 0.153 \ k/m \end{cases}$

$(\alpha = -0.18, \ 1.847)$

(b) Cont Similarly for $u = [1 \ -1 \ 0 \ \beta]^T$

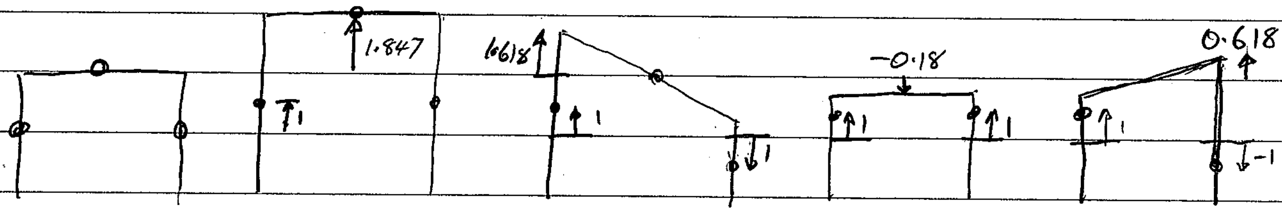
Row 1 of (1): $2k - \omega^2 m + ak\beta = 0 \Rightarrow (2 + a\beta)k = \omega^2 m$ — (4)

Row 4 of (1): $\cancel{a}ak + \cancel{a}a^2\beta(k - \omega^2 m) = 0 \Rightarrow (1 + a\beta)k = \omega^2 a\beta m$ — (5)

$$\frac{(4)}{(5)} \Rightarrow \frac{2 + a\beta}{1 + a\beta} = \frac{1}{a\beta} \Rightarrow a^2\beta^2 + a\beta - 1 = 0$$

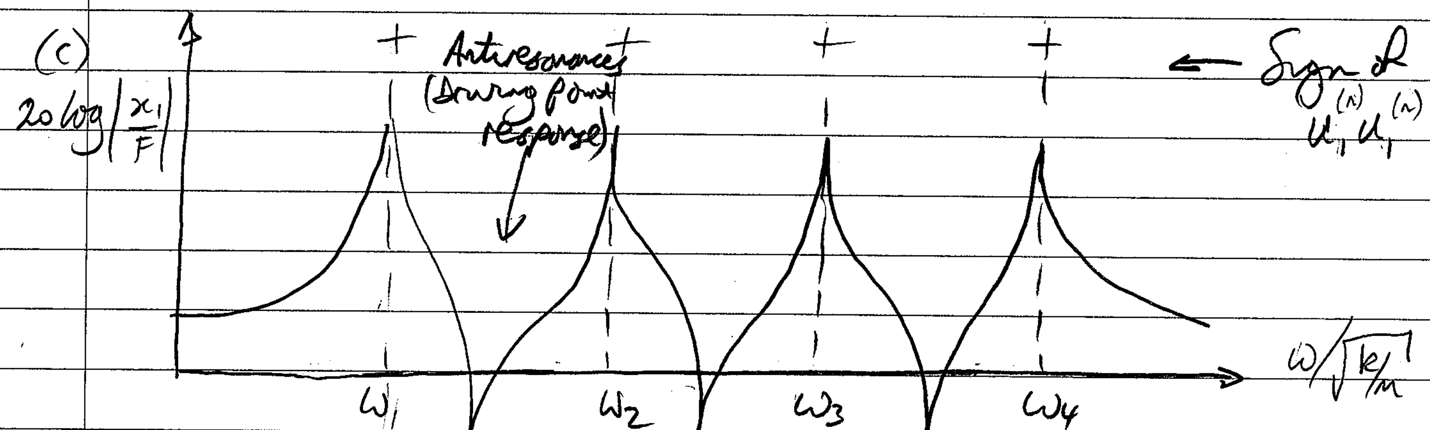
$$\Rightarrow \beta = \frac{-a \pm \sqrt{5a^2}}{2a^2} = \frac{-1 \pm \sqrt{5}}{2a} \doteq \begin{cases} 0.618/a \\ -1.618/a \end{cases}$$

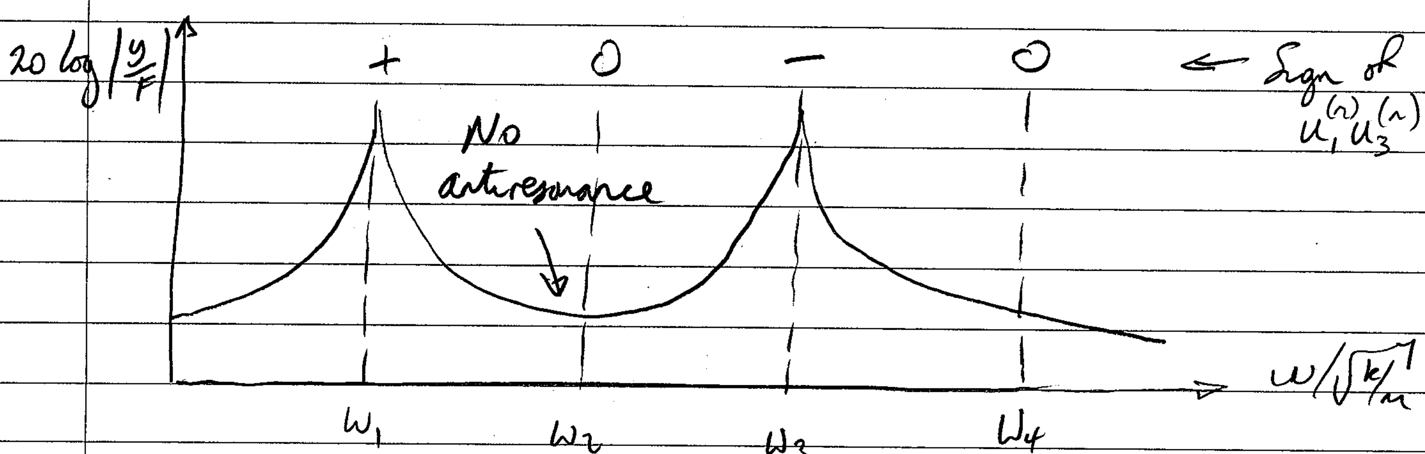
So (4) gives $$\omega^2_{2,4} = \left(\frac{3 \pm \sqrt{5}}{2}\right)k/m = \begin{cases} 0.382\ k/m \\ 2.618\ k/m \end{cases}$$



$\alpha = 1.847$, $\omega_1^2 = 0.153 k/m$ $\quad$ $a\beta = -1.618$, $\omega_2^2 = 0.382 k/m$ $\quad$ $\alpha = -0.18$, $\omega_3^2 = 2.18 k/m$ $\quad$ $a\beta = 0.618$, $\omega_4^2 = 2.618 k/m$

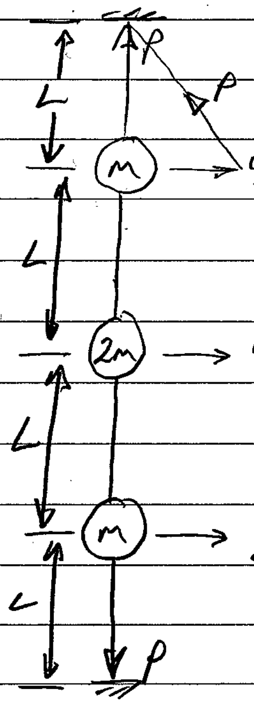
(c)





4 ~~3.~~

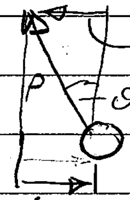
(a)



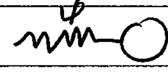
$\underline{KE}$ $\quad T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2}(2m)\dot{y}_2^2 + \frac{1}{2} m \dot{y}_3^2$

$$\Rightarrow [m] = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

Consider a string



$P\sin\theta \approx \frac{Py}{L} \Rightarrow$

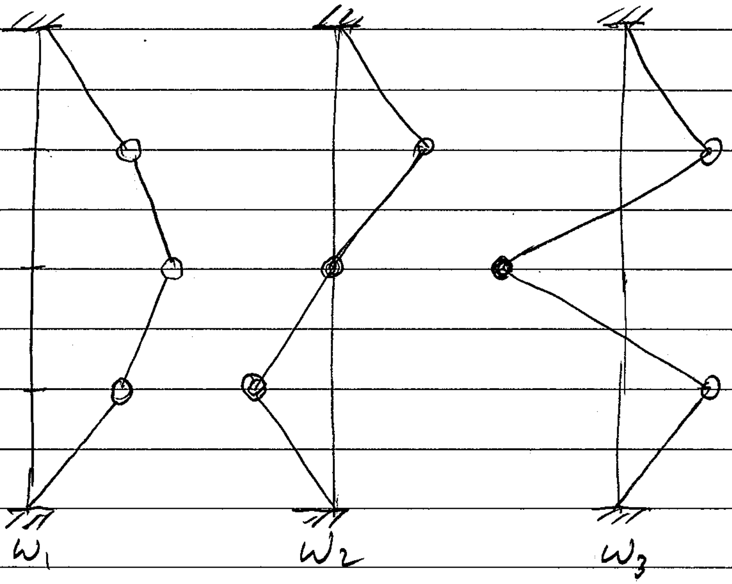


$$\therefore V = \frac{1}{2}(P/L) y_1^2 + \frac{1}{2}(P/L)(y_2 - y_1)^2 + \frac{1}{2}(P/L)(y_3 - y_2)^2 + \frac{1}{2}(P/L) y_3^2$$

$$= \frac{1}{2}(P/L)\left[2y_1^2 + 2y_2^2 + 2y_3^2 - 2y_1 y_2 - 2y_2 y_3\right]$$

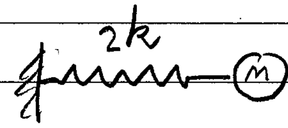
$$\therefore [k] = P/L \begin{array}{c} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \end{array} \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}$$

(b) Mode shapes



Mode 2, $y_2 = 0$

Equivalent system is



$$\Rightarrow \omega_2^2 = \frac{2k}{m} = \frac{2P}{Lm}$$

Exact

4~~3~~ (cont)

Eigenvalue problem is $([k]-\omega^2[m])\underline{u}=\underline{0}$

$$\begin{bmatrix} 2-\lambda^2 & -1 & 0 \\ -1 & 2-2\lambda^2 & -1 \\ 0 & -1 & 2-\lambda^2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \underline{0} \quad \text{with } \lambda^2 = \frac{\omega^2 L m}{P}$$

$$(2-\lambda^2)\left[(2-2\lambda^2)(2-\lambda^2)-1\right] + \left[-(2-\lambda^2)\right]$$

$$\Rightarrow 2(2-\lambda^2)(\lambda^4-3\lambda^2+1)=0 \quad \therefore \lambda_2^2 = 2 \;\&\; \lambda_{1,3}^2 = \frac{3\pm\sqrt{5}}{2}$$

$$\Rightarrow \omega^2 = 0.382\frac{P}{Lm}, \; \frac{2P}{Lm}, \; 2.62\frac{P}{Lm}$$

(c) Rayleigh $\omega_1^2 \le \frac{V_{max}}{T^*} = \frac{\frac{1}{2}P/L\left[2y_1^2+2y_2^2+2y_3^2-2y_1y_2-2y_2y_3\right]}{\frac{1}{2}m\left[y_1^2+1.8y_2^2+y_3^2\right]}$

<u>Method 1</u> – Guess first mode shape say $[1 \;\; 1.5 \;\; 1]^T$ (approximate) (or calculate first mode shape using λ_1)

– Use Rayleigh (above) to determine ω_1^2:

$$\omega_1^2 \approx \frac{P}{Lm}\frac{\left[2+2(1.5)^2+2-2(1.5)-2(1.5)\right]}{\left[1^2+1.8(1.5)^2+1^2\right]} = 0.413\frac{P}{Lm}$$

ie <u>4.04% increase in ω_1</u>

<u>Method 2 (exact)</u>

Assume mode shape is $[1 \;\; \alpha \;\; 1]^T$

$$\omega_1^2 = R = \frac{P}{Lm}\frac{\left[\cancel{2}+\cancel{2}\alpha^2+\cancel{2}-\cancel{2}\alpha-\cancel{2}\alpha\right]}{1+1.8\alpha^2+1} = \frac{2P}{Lm}\frac{(2+\alpha^2-2\alpha)}{2+1.8\alpha^2}$$

To find exact ω_1^2, $\frac{dR}{d\alpha}=0$:

$$(2+1.8\alpha^2)(2\alpha-2)-(2+\alpha^2-2\alpha)(3.6\alpha)=0$$

$$\Rightarrow \alpha = 1.584, -0.704 \Rightarrow \omega_1^2 = 0.4116\, P/Lm$$

ie 3.80% increase in 1st natural freq