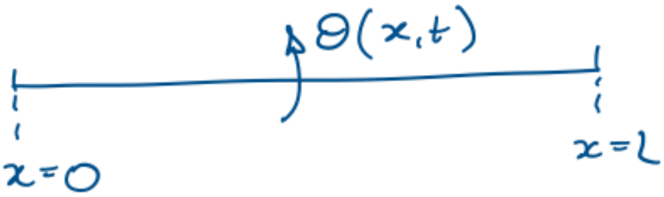
Solutions

Q1

(a)



for free-free shaft: $u_n(x) = \beta \cos kx$.

& $u'(L) = 0 \Rightarrow \sin kL = 0$.

$kL = n\pi$.

$$k_n = n\pi/L \Rightarrow \omega_n = k_n\sqrt{G/\rho} = \frac{n\pi}{L}\sqrt{G/\rho}$$

so $u_n(x) = \beta_n \cos n\pi x/L$ includes $n=0$

(b)

$$H(0,x_1,\omega) = \sum \frac{u_n(0)\,u_n(x_1)}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

mass-normalise: $\int u_n^2\,dm = 1$.

$dm = \rho A J\,dx$

$$\rho A J \int_0^L \beta_n^2 \cos^2 kx\,dx = 1.$$

$$\beta_n^2 \frac{\rho A J}{2}\int_0^L (1 + \cos 2kx)\,dx = 1$$

$$\beta_n^2 \frac{\rho A J}{2}\left[x + \frac{\sin 2n\pi x}{L}\right]_0^L = 1.$$

$\hookrightarrow 0.$

$$\beta_n^2 \rho \frac{AJ}{2} \cdot L = 1. \Rightarrow \beta_n = \sqrt{\frac{2}{\rho A L J}}$$

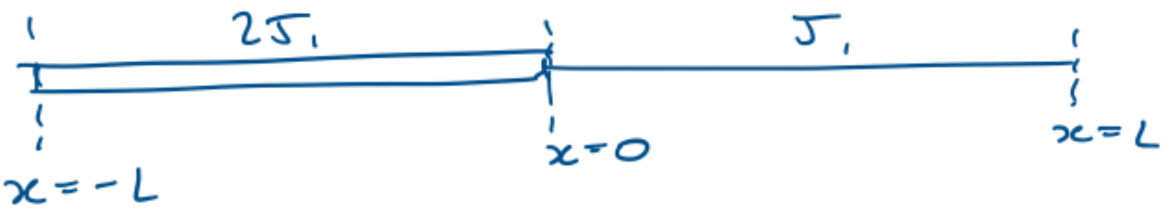
hence:
$$H(0, x_1, \omega) = \frac{2}{\rho A L J_1} \sum_{n=0}^{\infty} \frac{\cos \frac{n\pi x_1}{L}}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

note: $\cos 0 = 1$ hence only one cosine in numerator

note: answers with & without damping accepted.

$$\& \quad \omega_n = \frac{n\pi}{L}\sqrt{G/\rho}$$

(c)



(i) $$H_c = \frac{H_1 H_2}{H_1 + H_2}$$

H_2 = as above but with $J_2 = 2J_1$

so $H_2 = H_1/2$.

$$\Rightarrow H_c = \frac{H_1^2/2}{H_1 + H_1/2} = \frac{H_1}{3} \quad \left(\text{ie } \lambda = \frac{1}{3}\right)$$

(ii) Compatibility: $\theta_1(0,t) = \theta_2(0,t)$ //

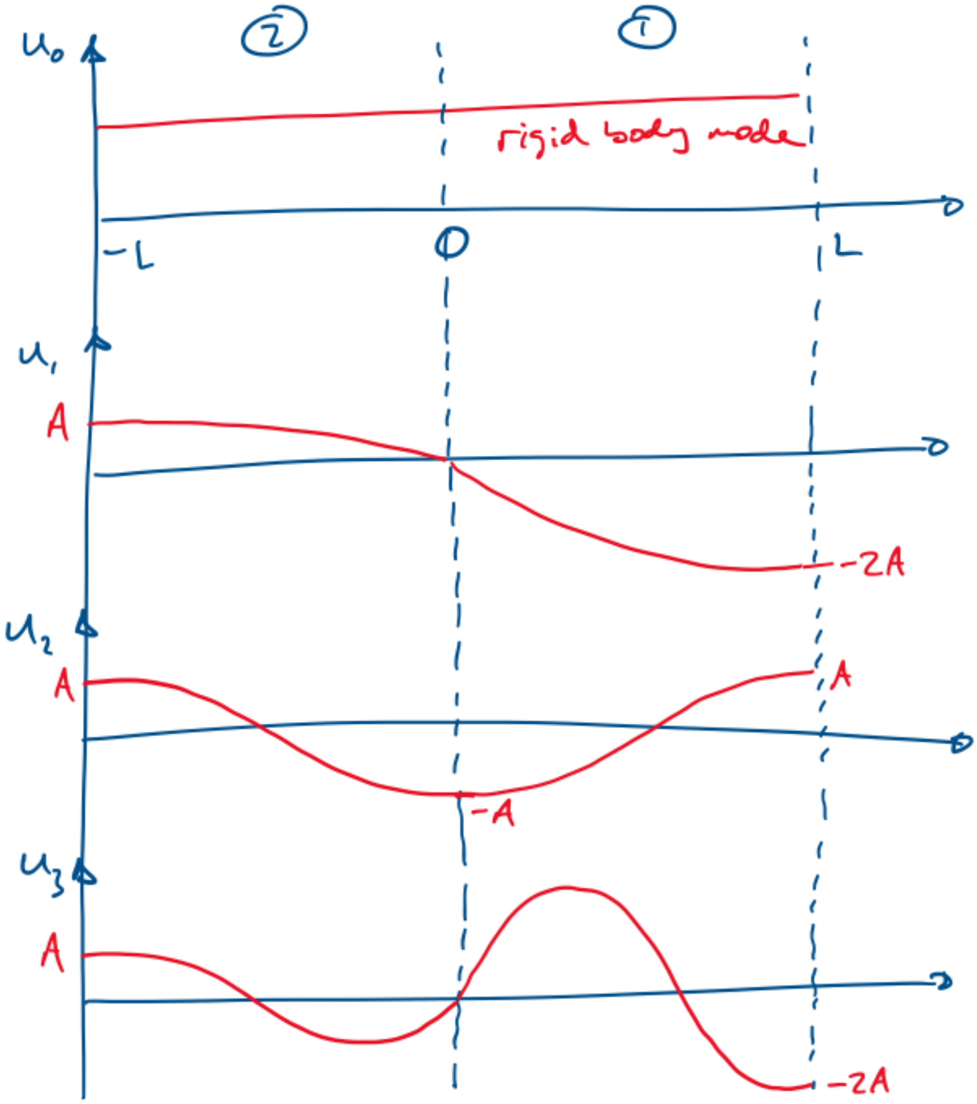
Torque balance: $GJ_1\theta_1'\big|_{x=0} = GJ_2\theta_2'\big|_{x=0}$.

ie $\theta_1'\big|_{x=0} = 2\,\theta_2'\big|_{x=0}$ //

- $H_c = H_1/3$ so ω_n are same for coupled shaft: & free-free mode shapes satisfy coupling conditions. Ie cosine mode shapes.

- Any modes not visible in H_c must have node @ $x=0$. Ie sine mode shapes. To satisfy torque balance:

$$\omega_n^{(2)} = \omega_n^{(1)}/2$$



$k_0 = 0$
$\omega_0 = 0.$

$\lambda = 4L, \quad k = \frac{2\pi}{4L} = \frac{\pi}{2L}$
$\omega_n = \frac{\pi}{2L}\sqrt{G/\rho}$

$\lambda = 2L, \quad k = \frac{\pi}{L}$
$\omega_n = \frac{\pi}{L}\sqrt{G/\rho}$

$\lambda = 4L/3, \quad k = \frac{3\pi}{2L}$
$\omega_n = \frac{3\pi}{2L}\sqrt{G/\rho}$

(iii) Characteristic admittance $z \equiv \dfrac{\dot{\theta}_F}{-GJ\theta_F'}$

$$\theta_F = \theta_0 e^{i(kx-\omega t)}$$

$$\dot{\theta}_F = -i\omega\,\theta_F$$

$$-GJ\,\theta_F' = -ik\,\theta_F$$

$$z = \frac{\omega}{GJk} = \frac{c}{GJ} = \frac{1}{GJ}\cdot\sqrt{G/\rho} = \frac{1}{J\sqrt{\rho G}}$$

junction
2 → 1

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{1/J_1 - 1/J_2}{1/J_1 + 1/J_2} = \frac{J_2 - J_1}{J_2 + J_1}$$

$$J_2 = 2J_1$$

$$\Rightarrow \quad R = \frac{J_1}{3J_1} = 1/3$$

R not close to 0 or 1 so coupling is strong.

Q2

(a) (i) $\rho A \ddot{y} + EI y^{iv} = 0$

let $y = U e^{i\omega t}$

$\Rightarrow \quad EI U^{iv} - \rho A \omega^2 U = 0.$

$$U = A\cos kx + B\sin kx + C\cosh kx + D\sinh kx$$
$$U' = -A\sin + B\cos + C\sinh + D\cosh \quad \times k$$
$$U'' = -A\cos - B\sin + C\cosh + D\sinh \quad \times k^2$$
$$U''' = A\sin - B\cos + C\sinh + D\cosh \quad \times k^3$$

BC's: @ $x = 0$: $U = U'' = 0.$

$\Rightarrow \quad A + C = 0.$

& $A - C = 0. \quad \Rightarrow \quad A = C = 0.$

@ $x = L$: $U'' = U''' = 0.$

$B\sin kL = D\sinh kL$ —— ①

$B\cos kL = D\cosh kL$ —— ②

$\frac{①}{②} \Rightarrow \tan kL = \tanh kL$

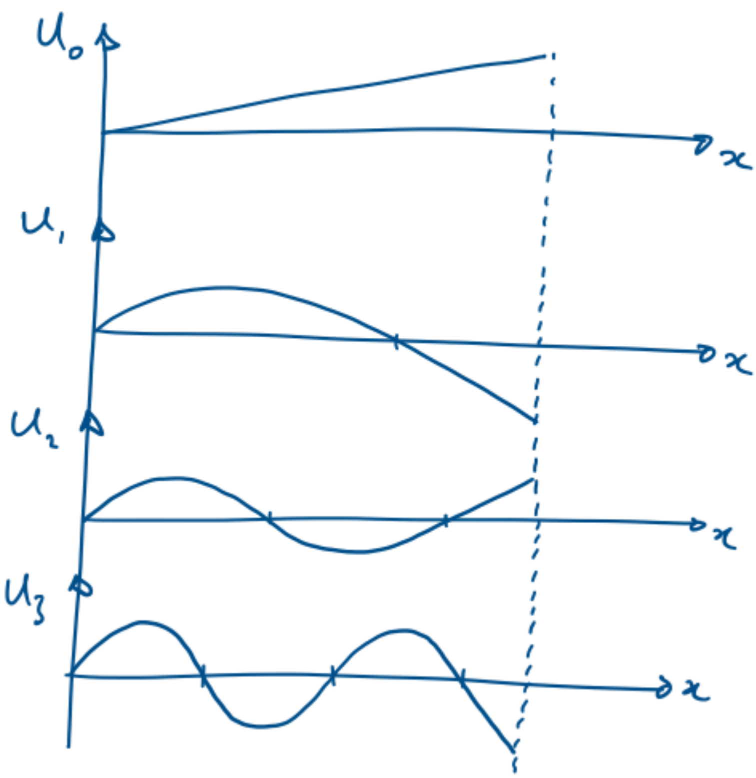
(ii) for $n=0$: $U_0(x) = x$ (rigid body mode)

for $n \geq 1$: $U_n(x) = B\sin k_n x + D\sinh k_n x$

$$U_n(x) = B_n\left[\sin k_n x + \frac{\sin k_n L}{\sinh k_n L}\sinh k_n x\right]$$

from ①, or $\frac{\cos k L}{\cosh k L}$ from ②

both same for k_n



(b) (i) $U_F = U_0 e^{i(kx - \omega t)}$

$\rightarrow$ PDE $\Rightarrow$ $-\rho A\omega^2 + EIk^4 + Pk^2 = 0$.

$$\omega^2 = \frac{EIk^4 + Pk^2}{\rho A} = \frac{k^2(EIk^2 + P)}{\rho A}$$

$$\omega = k\sqrt{\frac{EIk^2 + P}{\rho A}}$$ —— dispersion equation

$$c_p = \frac{\omega}{k} = \sqrt{\frac{EIk^2 + P}{\rho A}}$$

$$c_g = \frac{\partial \omega}{\partial k} = \frac{4EIk^3 + 2Pk}{\rho A} \cdot \frac{1}{2k\sqrt{\frac{EIk^2+P}{\rho A}}}$$

$$= \frac{2EIk^2 + P}{\sqrt{\rho A\left(EIk^2 + P\right)}}$$

(iii) $R = \frac{V}{\tilde{T}}$

$$V = \underbrace{\frac{1}{2} P \int y'^2 \, dx}_{V'} + \underbrace{\frac{1}{2} EI \int y''^2 \, dx}_{V_0}$$

$$\tilde{T} = \frac{1}{2} \rho A \int y^2 \, dx = T_0 .$$ unchanged by tension.

$$R = \frac{V_0\left(1 + \frac{\frac{1}{2} P \int y'^2 \, dx}{V_0}\right)}{T_0} = \omega_n^2\left(1 + \frac{\frac{1}{2} P \int y'^2 \, dx}{V_0}\right)$$

$$\Delta \simeq \frac{1}{2} \frac{V'}{V_0}$$

from square root.

$$y = \sin k_n x$$

$$y' = k_n \cos k_n x.$$

$$y'^2 = k_n^2 \cos^2 k_n x = \frac{1}{2} k_n^2 \left(1 + \cos 2k_n x\right)$$

$$\frac{1}{2} P \int_0^L y'^2 \, dx = \frac{1}{2} P \cdot \frac{1}{2} k_n^2 \int_0^L \left(1 + \cos 2k_n x\right) dx$$

$$= \frac{1}{4} P k_n^2 \left[x + \frac{1}{2k_n} \underbrace{\sin 2k_n x}_{} \right]_0^L$$

$$\sin 2k_n L \simeq 1$$

$$= \frac{1}{4} P k_n^2 L \left[L + \frac{1}{2k} \right]$$

$$= \frac{1}{4} P (kL)^2 \left[1 + \frac{1}{2kL} \right]$$

$$V_0 = \frac{1}{2} EI \int_0^L y''^2 \, dx = \frac{1}{2} EI \int_0^L k_n^2 \sin^2 k_n x \, dx$$

$$= \frac{1}{2} EI \cdot \frac{1}{2} k_n^4 \int_0^L \left(1 - \cos 2k_n x\right) dx$$

$$= \frac{1}{4} EI k_n^4 \left[x - \frac{1}{2k_n} \underbrace{\sin 2k_n x}_{} \right]_0^L$$

$$\sin 2k_n L \simeq 1.$$

$$= \frac{1}{4} EI k_n^4 L \left[L - \frac{1}{2k_n} \right]$$

$$= \frac{1}{4} EI k_n^4 L^2 \left[1 - \frac{1}{2kL} \right]$$

$$\frac{v'}{v_0} = \frac{PL^2}{EI(kL)^2} \frac{\left(1 + \frac{1}{2}kL\right)}{\left(1 - \frac{1}{2}kL\right)}$$

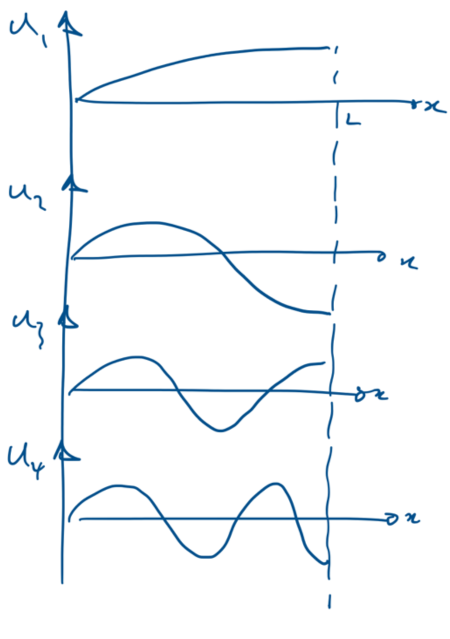
$$\simeq 0.084 \cdot \frac{PL^2}{EI}$$

$$\text{so} \quad \Delta \simeq 0.04 \frac{PL^2}{EI} \quad //$$

(iii) If P large (specifically if $P \gg EIk^2$) then tension dominates bending stiffness & the beam behaves as a string. The boundary conditions would become a fixed-free string, so the mode shapes would be:

$$u_n = \sin k_n x \,, \qquad k_n = (n - \tfrac{1}{2})\pi/L \,.$$

$$\& \quad \omega_n = (n - \tfrac{1}{2})\pi/L \cdot \sqrt{P/\rho A} \quad //$$



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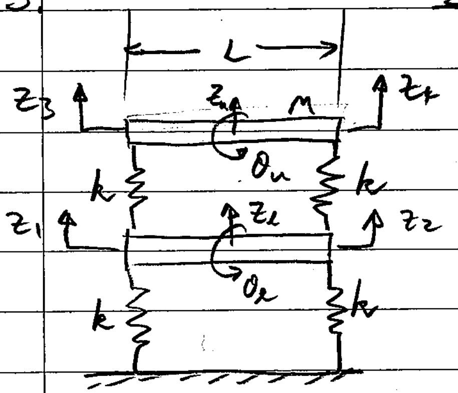


$$T = \tfrac{1}{2} m \dot{z}_l^2 + \tfrac{1}{2} m \dot{z}_u^2 + \tfrac{1}{2} I \dot{\theta}_l^2 + \tfrac{1}{2} I \dot{\theta}_u^2$$

$$\dot{z}_l = \frac{\dot{z}_1 + \dot{z}_2}{2} \qquad \dot{z}_u = \frac{\dot{z}_3 + \dot{z}_4}{2}$$

$$\theta_l = \frac{\dot{z}_2 - \dot{z}_1}{l} \qquad \dot{\theta}_u = \frac{\dot{z}_4 - \dot{z}_3}{l}$$

So $$T = \frac{1}{2} m \left[\left(\frac{\dot{z}_1 + \dot{z}_2}{2}\right)^2 + \left(\frac{\dot{z}_3 + \dot{z}_4}{2}\right)^2 \right] + \frac{1}{2}\cdot\frac{1}{12} m l^2 \left[\left(\frac{\dot{z}_2 - \dot{z}_1}{l}\right)^2 + \left(\frac{\dot{z}_4 - \dot{z}_3}{l}\right)^2 \right]$$

$$= \frac{m}{8}\left[\dot{z}_1^2 + \dot{z}_2^2 + 2\dot{z}_1\dot{z}_2 + \dot{z}_3^2 + \dot{z}_4^2 + 2\dot{z}_3\dot{z}_4 \right]$$

$$+ \frac{m}{24}\left[\dot{z}_2^2 + \dot{z}_1^2 - 2\dot{z}_1\dot{z}_2 + \dot{z}_4^2 + \dot{z}_3^2 - 2\dot{z}_4\dot{z}_3 \right]$$

$$= \frac{m}{2}\left[\frac{\dot{z}_1^2}{3} + \frac{\dot{z}_2^2}{3} + \frac{\dot{z}_3^2}{3} + \frac{\dot{z}_4^2}{3} - \frac{\dot{z}_1\dot{z}_2}{3} - \frac{\dot{z}_3\dot{z}_4}{3} \right]$$

$$= \frac{m}{2} \begin{bmatrix} \dot{z}_1 & \dot{z}_2 & \dot{z}_3 & \dot{z}_4 \end{bmatrix} \begin{bmatrix} 1/3 & -1/6 & \multicolumn{2}{c}{[0]} \\ -1/6 & 1/3 & & \\ \multicolumn{2}{c}{[0]} & 1/3 & -1/6 \\ & & -1/6 & 1/3 \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{Bmatrix}$$

$$V = \tfrac{1}{2} k \left[z_1^2 + z_2^2 + (z_3 - z_1)^2 + (z_4 - z_2)^2 \right]$$

$$= \tfrac{1}{2} k \left[2z_1^2 + 2z_2^2 + z_3^2 + z_4^2 - 2z_1z_3 - 2z_2z_4 \right]$$

$$= \frac{k}{2} \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix}$$

Eigenvalues and vectors:

$$\left([K] - \omega^2 [M]\right) \underset{\sim}{u} = [0]$$

$$\left(k\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} - \omega^2 m \begin{bmatrix} 1/3 & -1/6 & 0 & 0 \\ -1/6 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & -1/6 \\ 0 & 0 & -1/6 & 1/3 \end{bmatrix} \right) \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} = \underset{\sim}{0}$$

$$\begin{bmatrix} 2k - \omega^2 m/3 & +\frac{\omega^2 m}{6} & -k & 0 \\ \omega^2 m/6 & 2k - \omega^2 m/3 & 0 & -k \\ -k & 0 & k - \omega^2 m/3 & \omega^2 m/6 \\ 0 & -k & \omega^2 m/6 & k - \omega^2 m/3 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} = \underset{\sim}{0} \quad (1)$$

Modes 1 & 3

Assume 'bounce' mode $[1 \quad 1 \quad \alpha \quad \alpha]^T$

Row 1 $\quad 2k - \frac{\omega^2 m}{3} + \frac{\omega^2 m}{6} - k\alpha = 0$

ie $\quad k(2-\alpha) - \frac{\omega^2 m}{6} = 0 \quad$ —— (2)

Row 3 $\quad -k + \alpha\left(k - \omega^2 m/3 + \omega^2 m/6\right) = 0$

ie $\quad k(\alpha - 1) - \alpha \frac{\omega^2 m}{6} = 0. \quad$ —— (3)

Eliminate ω^2 between (3) & (2)

(2) $\rightarrow \quad \cancel{k}(\alpha-1) - \frac{\alpha \cancel{m}}{\cancel{6}} \frac{\cancel{6}\cancel{k}(2-\alpha)}{\cancel{m}} = 0$

$$\alpha - 1 - 2\alpha + \alpha^2 = 0 \rightarrow \alpha^2 - \alpha - 1 = 0$$

$$\Rightarrow \alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = 1.618, -0.618$$

$$② \rightarrow \omega^2 = \frac{6k}{m}(2-\alpha)$$

If $\alpha = \dfrac{1+\sqrt{5}}{2}$ $\quad\quad \omega^2 = \dfrac{6k}{m}\left(2 - \dfrac{(1+\sqrt{5})}{2}\right)$

$\underset{\sim}{u}^{(2)} = \begin{Bmatrix} 1 \\ 1 \\ 1.618 \\ 1.618 \end{Bmatrix}$ $\quad\quad = \dfrac{6k}{m}\left(\dfrac{4-1-\sqrt{5}}{2}\right)$

$$= \frac{6k}{m}\left(\frac{3-\sqrt{5}}{2}\right)$$

$$= 2.292 \checkmark$$

$\alpha = \dfrac{1-\sqrt{5}}{2}$ $\quad\quad \omega^2 = \dfrac{6k}{m}\left(2 - \dfrac{(1-\sqrt{5})}{2}\right)$

$u^{(4)} = \begin{Bmatrix} 1 \\ 1 \\ -0.618 \\ -0.618 \end{Bmatrix}$ $\quad\quad = \dfrac{6}{2}\left(4-1+\sqrt{5}\right)$

$$= 3(3+\sqrt{5}) = 15.708 \checkmark$$

Assume $\quad \underset{\sim}{u} = [1 \quad -1 \quad \beta \quad -\beta]^T$

Pitch modes

Row (1) of (1): $\quad 2k - \dfrac{\omega^2 M}{3} - \dfrac{\omega^2 M}{6} - k\beta = 0$

ie $\quad k(2-\beta) - \omega^2 M/2 = 0 \quad$ —— ④

Row (3) of (1): $\quad -k + (k - \omega^2 M/3)\beta - (\omega^2 M/6)\beta = 0$

$$k(\beta-1) - \frac{\omega^2 M}{2}\beta = 0 \quad \text{—— ⑤}$$

④ & ⑤ $\rightarrow$ $\cancel{k}(\beta-1) - \cancel{\frac{M}{2}}\beta \frac{2\cancel{k}(2-\beta)}{\cancel{m}} = 0$

$$\beta - 1 - 2\beta + \beta^2 = 0 \longrightarrow \beta^2 - \beta - 1 = 0.$$

$$\beta = \frac{1 \pm \sqrt{5}}{2}$$

④ $\rightarrow \quad \omega^2 = \dfrac{2k(2-\beta)}{m}$

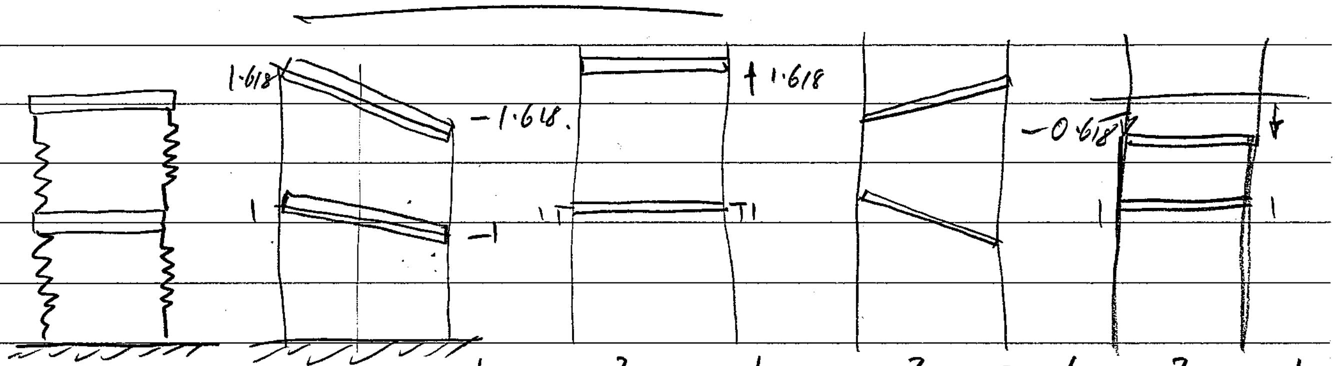
$\begin{Bmatrix} 1 \\ -1 \\ 1.618 \\ -1.618 \end{Bmatrix}$ $\quad \beta = \dfrac{1+\sqrt{5}}{2} = 1.618 \longrightarrow \omega_1^2 = \dfrac{2k}{m}\left(2 - \dfrac{1+\sqrt{5}}{2}\right)$

$= \dfrac{\cancel{2}k}{m}\left(\dfrac{3-\sqrt{5}}{\cancel{2}}\right) = 0.764\, k/m$ ✓

$\begin{Bmatrix} 1 \\ -1 \\ -0.618 \\ 0.618 \end{Bmatrix}$ $\quad \beta = \dfrac{1-\sqrt{5}}{2} = -0.618 \longrightarrow \omega_3^2 = \dfrac{2k}{m}\left(2 - \left(\dfrac{1-\sqrt{5}}{2}\right)\right)$

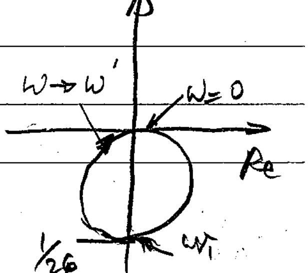
$= \cancel{2}k/m\left(\dfrac{3+\sqrt{5}}{\cancel{2}}\right)$

$= 5.236\, k/m$ ✓



$\omega_1^2 = 0.764\, k/m \qquad \omega_2^2 = 2.292\, k/m \qquad \omega_3^2 = 5.236\, k/m \qquad \omega_4^2 = 15.71\, k/m$

Modal Constant $\qquad + \qquad + \qquad - \qquad -$



4 (a)

$$T = \tfrac{1}{2} m \left[\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2 + \dot{x}_5^2\right]$$

$$V = \tfrac{1}{2} k \left[(x_2-x_1)^2 + (x_3-x_2)^2 + (x_4-x_3)^2\right] + \tfrac{1}{2} s (x_5 - x_4)^2$$

(b) Rayleigh's quotient is:

$$\omega^2 = R = \frac{V}{T^*} = \frac{\tfrac{1}{2} k \left[(x_2-x_1)^2 + (x_3-x_2)^2 + (x_4-x_3)^2\right] + \tfrac{1}{2} s (x_5-x_4)^2}{\tfrac{1}{2} m \left[x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\right]} \quad \text{——} (1)$$

Lowest frequency is for rigid body mode $\underline{q} = [1\,1\,1\,1\,1]^T$

then $\omega^2 = \dfrac{k\left[0^2 + 0^2 + 0^2\right] + s\left[0^2\right]}{m\left[5 \times 1^2\right]} = \underline{\underline{0}}$ ✓

(c) When $s = k$, the system is symmetric

For $q = [1 \;\; \alpha \;\; 0 \;\; -\alpha \;\; -1]^T$:

$$\omega^2 = R = \frac{\cancel{\tfrac{1}{2}} k \left[(\alpha-1)^2 + (0-\alpha)^2 + (-\alpha-0)^2 + (-1+\alpha)^2\right]}{\cancel{\tfrac{1}{2}} m \left[1^2 + \alpha^2 + 0^2 + \alpha^2 + 1^2\right]}$$

$$= \frac{k}{m} \frac{\cancel{2}\left[(1-\alpha)^2 + \alpha^2\right]}{\cancel{2}\left[1+\alpha^2\right]} = \frac{k}{m} \frac{(2\alpha^2 - 2\alpha + 1)}{1+\alpha^2} \quad \text{——} (2)$$

The exact mode shape is found by minimising R

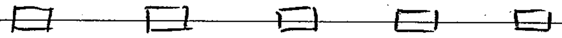
$$\frac{dR}{d\alpha} = \frac{(1+\alpha^2)(4\alpha-2) - (2\alpha^2 - 2\alpha + 1)(2\alpha)}{(1+\alpha^2)^2} \equiv 0$$

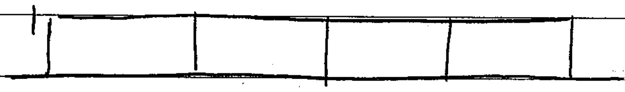
$$\Rightarrow \quad \alpha^2 + \alpha - 1 = 0$$

$$\Rightarrow \quad \alpha = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} = 0.618, \; -1.618$$

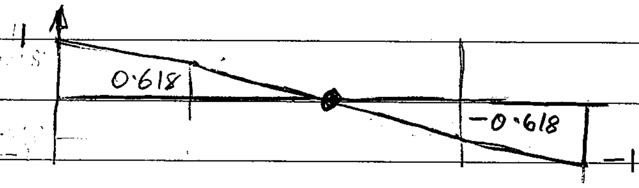
& (2) $\Rightarrow$ $\omega_2^2 = 0.3819 \dfrac{k}{m}$ & $\omega_4^2 = 2.618 \dfrac{k}{m}$

4 (c) Cont.

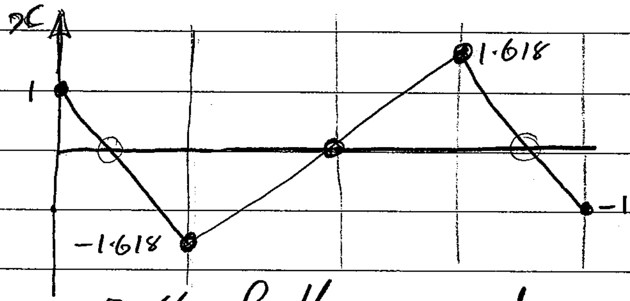




Mode 1, $\omega_1 = 0$, rigid body



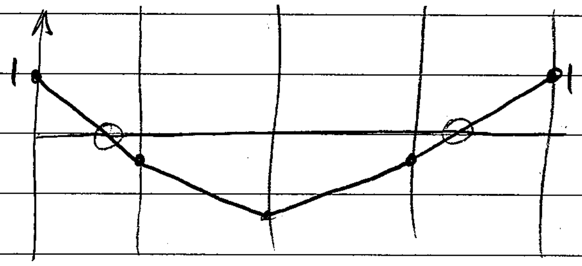
Mode 2, $\alpha = 0{\cdot}618$, $\omega_2 = 0{\cdot}618\sqrt{k/m}$



Mode 4, $\alpha = -1{\cdot}618$, $\omega_4 = 1{\cdot}618\sqrt{k/m}$

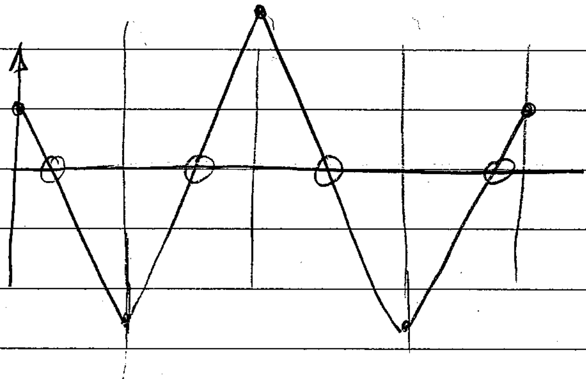
Both of these modes satisfy the shape $[1\ \alpha\ 0\ -\alpha\ -1]^T$ and they are therefore both <u>exact</u> <u>antisymmetric</u> modes

(d) Other 2 modes are <u>symmetric</u> with 2 and 4 'nodes'



Mode 3 2 nodes

$0{\cdot}618\sqrt{k/m} < \omega_3 < 1{\cdot}618\sqrt{k/m}$



Mode 5 4 nodes

$\omega_5 > 1{\cdot}618\sqrt{k/m}$

4 (e)

For a small change in S, ie S = 1.1k, the original mode shape is still a reasonable approximation to the correct mode. Rayleigh will therefore give a good estimate of the frequency

$$① \Rightarrow \omega^2 = \frac{\frac{1}{2}k\left[(\alpha-1)^2 + (0-\alpha)^2 + (-\alpha-0)^2\right] + \frac{1}{2}1.1k(-1+\alpha)^2}{\frac{1}{2}m\left[2(1+\alpha)^2\right]}$$

$$= \underbrace{\frac{\frac{1}{2}k\left[(\alpha-1)^2 + 2\alpha^2 + (\alpha-1)^2\right]}{\frac{1}{2}m\left[2(1+\alpha^2)\right]}}_{\substack{\omega_1^2 \\ \text{from } (2)}} + \underbrace{\frac{\cancel{\tfrac{1}{2}}(0.1k)(\alpha-1)^2}{\cancel{\tfrac{1}{2}}m\left[2(1+\alpha^2)\right]}}_{\Delta\omega^2}$$

for $\alpha = 0.618$, $\Delta\omega^2 = 0.00528\ k/m$

So $\omega^2 = (0.3819 + 0.00528)k/m = 0.3872\ k/m$

Hence % increase is $\dfrac{\sqrt{0.3872} - 0.618}{0.618} = 0.686\ \%$