EGT2
ENGINEERING TRIPOS PART IIA

Friday 28 April 2023   9.30 to 11.10

Module 3C6

VIBRATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Attachment: 3C5 Dynamics and 3C6 Vibration data sheet 2023 (7 pages).

Engineering Data Book.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A uniform 'free-free' shaft of circular cross section has polar moment of area J_1 and is made of a material with density ρ and shear modulus G. The axial distance along the shaft is x. It is driven by a harmonically varying torque about the shaft axis $T(0,t) = T_0 e^{i\omega t}$ at one end ($x = 0$) and is free at the other ($x = L$).

(a) (i) Find the natural frequencies and mode shapes for free vibration of the shaft. [10%]

(ii) Find a summation expression for the transfer function $H(0, x_1, \omega)$ from input torque at $x = 0$ to output angular displacement at $x = x_1$, where x_1 is an arbitrary distance along the shaft. [20%]

(b) A second circular shaft, made of the same material, is connected to the first at $x = 0$ as illustrated in Fig. 1. The second shaft has length L and polar moment of area $J_2 = 2J_1$.

(i) Show that the driving point transfer function H_C of the coupled shafts at the connection point is proportional to $H_1 \equiv H(0,0,\omega)$, i.e. show that $H_C = \lambda H_1$, and find the constant of proportionality λ. [10%]

(ii) By considering H_C and the conditions at the connection point, sketch the first four mode shapes of the coupled system and find the corresponding natural frequencies. [40%]

(iii) Find the reflection coefficient R at $x = 0$ for a wave travelling in the positive x direction. Comment on the strength of coupling of the shafts. [20%]

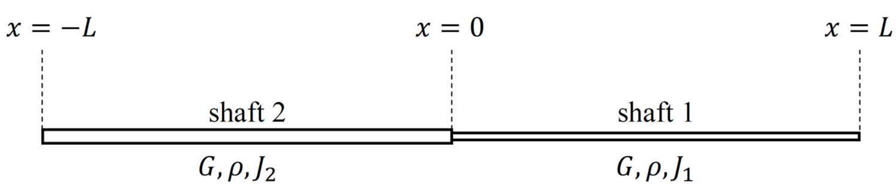

Fig. 1

2 A beam of length L, uniform cross-section of area A and second moment of area I is made from a material with density ρ and Young's modulus E. The beam undergoes small-amplitude transverse vibration $y(x,t)$.

(a) The beam is pinned at one end ($x = 0$) and is free at the other ($x = L$).

(i) Starting from the governing equation for transverse vibration of a beam, derive an expression whose solutions give the wavenumbers k_n for the modes of the beam. [20%]

(ii) Find expressions for the mode shapes $u_n(x)$ in terms of k_n and L and sketch the first four mode shapes. [20%]

(b) A constant horizontal force is applied to the free end of the beam so that it has a tension P, as illustrated in Fig. 2. The governing equation for free vibration of a beam under tension P is given by:

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} - P \frac{\partial^2 y}{\partial x^2} = 0$$

(i) Find expressions for the phase velocity c_p and the group velocity c_g of the tensioned beam as a function of wavenumber k. [20%]

(ii) Use Rayleigh's principle to estimate the natural frequency ω_1' of the first bending mode of the beam under tension. Use $u_1(x) = \sin(3.9266\, x/L)$ as an approximation to the mode shape. Write your answer in the form $\omega_1' = \omega_1(1 + \Delta)$, where ω_1 is the frequency of the first bending mode without tension and Δ is to be found. You may use the result that the potential energy of a beam under tension is given by

$$V = \frac{1}{2} EI \int \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx + \frac{1}{2} P \int \left(\frac{\partial y}{\partial x}\right)^2 dx$$

[30%]

(iii) What do you expect will happen to the mode shapes and natural frequencies if the tension P is high? [10%]

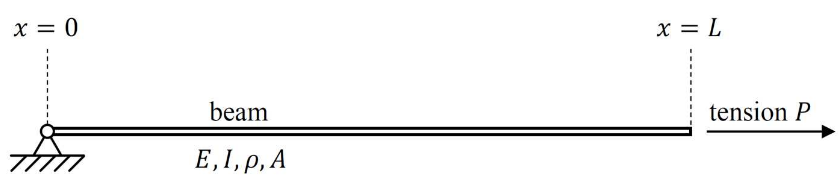

Fig. 2

3 Figure 3 shows a simplified two-dimensional model of the vertical vibration of a building with 2 floors. The floors are assumed to be rigid beams of mass m and length l, connected by linear springs of stiffness k, representing the vertical stiffnesses of the walls. The floors are constrained to move vertically and the displacements of the ends of the floors are z_1 to z_4 as shown.

(a) Assuming small motions, write down expressions for the kinetic and potential energies T and V in terms of the coordinates z_1 to z_4. Hence or otherwise write down the mass matrix and show that the stiffness matrix can be written as:

$$[K] = k\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

where the vector of generalised coordinates is $[z_1 \quad z_2 \quad z_3 \quad z_4]^T$. [30%]

(b) The system has two natural modes with eigenvectors of the form $[1 \quad 1 \quad \alpha \quad \alpha]^T$. By substituting these mode shapes into the eigenvalue calculation, or otherwise, determine the value of α and determine the natural frequencies. Sketch all four mode shapes. [40%]

(c) Assuming that the modes are lightly damped, sketch a log amplitude plot and a phase plot for the transfer function describing the vertical displacement z_B of point B on the upper floor when a sinusoidal force f_A is applied to point A on the lower floor, as shown in the figure, where A and B are in the right half of the beams. [20%]

(d) Sketch a polar plot (ie Imaginary part vs Real part) of the transfer function in part (c), in the vicinity of the mode with the lowest natural frequency. Show salient values. [10%]

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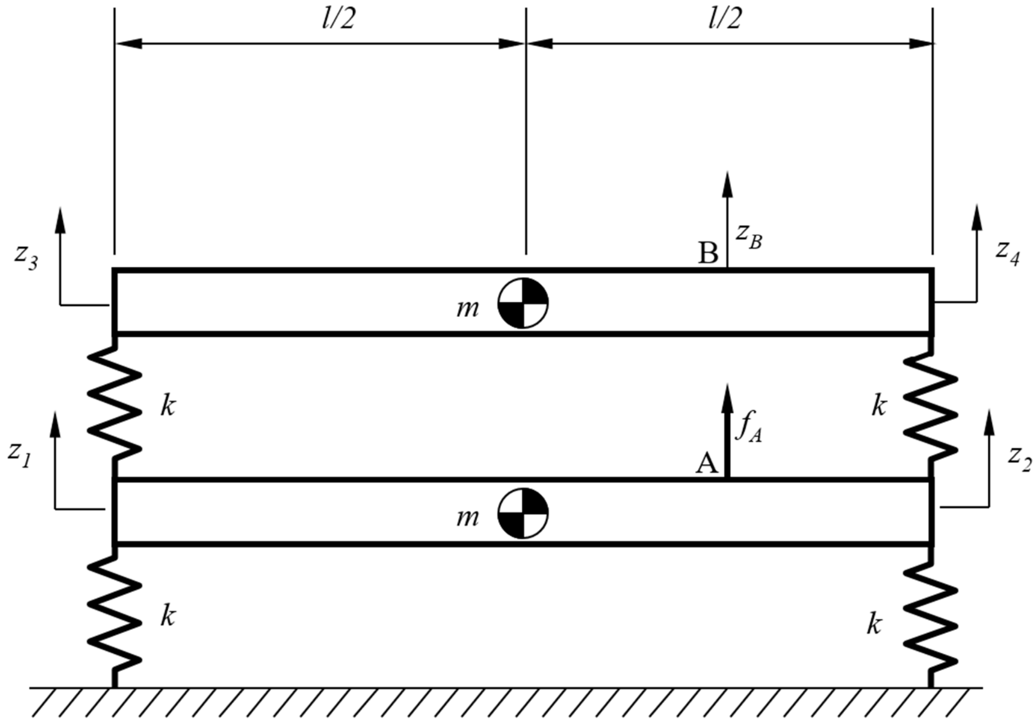

Fig. 3

(TURN OVER

4. Five heavy lorries of mass m are connected together to form a 'platoon' as shown in Fig. 4. Each lorry has a sensor to measure the distance to the vehicle in front and a proportional feedback control system that maintains a near constant spacing (or 'headway') between the vehicles. For considering the 'string stability' of the platoon, each headway can be modelled as a linear spring and the steady-state velocity of the platoon can be neglected. Four of the headway 'springs' have stiffness k and one has stiffness S. The longitudinal positions of the vehicles are defined by the vector of small displacements from their nominal positions $[x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T$.

(a) Write expressions for the kinetic and potential energies of the system. [10%]

(b) Using the results from part (a), write an expression for Rayleigh's quotient. Show that the quotient gives the correct natural frequency for the rigid body mode. [10%]

(c) For the case where $S = k$, the mode with the smallest non-zero natural frequency has the form $[1 \quad \alpha \quad 0 \quad -\alpha \quad -1]^T$. By differentiating Rayleigh's quotient, find the natural frequency. Sketch the corresponding mode shape. Discuss and sketch any other modes found by this analysis. [40%]

(d) Sketch the remaining mode shapes and indicate whether their corresponding natural frequencies are higher or lower than the frequencies you found in Part (c). [20%]

(d) If the stiffness of spring S is increased by 10%, ($S = 1.1k$), estimate the percentage change in the smallest non-zero natural frequency. [20%]

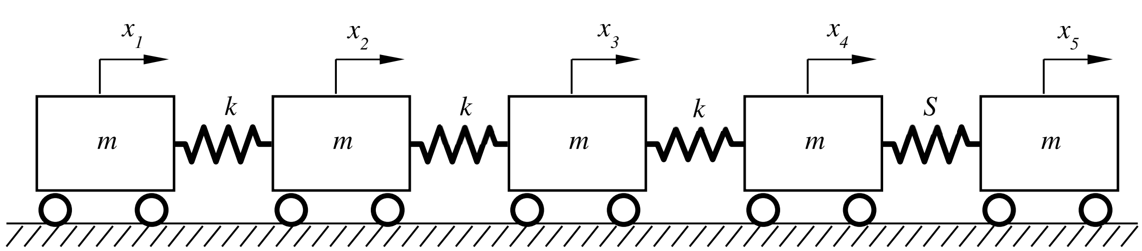

Fig. 4

END OF PAPER

Part IIA Data Sheet

Module 3C5 Dynamics
Module 3C6 Vibration

1 Dynamics in three dimensions

1.1 Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\boldsymbol{v}_\mathrm{G}$, M is the total mass, $\boldsymbol{v}_\mathrm{G}$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\begin{aligned}\boldsymbol{Q}^{(e)} &= (\boldsymbol{r}_\mathrm{G} - \boldsymbol{r}_\mathrm{P}) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_\mathrm{G} \\ &= \dot{\boldsymbol{h}}_\mathrm{P} + \dot{\boldsymbol{r}}_\mathrm{P} \times \boldsymbol{p}\end{aligned}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here $\boldsymbol{h}_\mathrm{P}$ and $\boldsymbol{h}_\mathrm{G}$ are the moments of momentum about P and G respectively, so that for example

$$\begin{aligned}\boldsymbol{h}_P &= \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_\mathrm{P}) \times m_i \dot{\boldsymbol{r}}_i \\ &= \boldsymbol{h}_\mathrm{G} + (\boldsymbol{r}_\mathrm{G} - \boldsymbol{r}_\mathrm{P}) \times \boldsymbol{p}\end{aligned}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) dm = \boldsymbol{I}\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $A = \int (y^2 + z^2) dm \qquad B = \int (z^2 + x^2) dm \qquad C = \int (x^2 + y^2) dm$

$D = \int yz\ dm \qquad E = \int zx\ dm \qquad F = \int xy\ dm$

where all integrals are taken over the volume of the body.

1.2 Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1.1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

1.3 Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= Q_1 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= Q_2 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= Q_3 \end{aligned}$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the"Gyroscope equations"):

$$\begin{aligned} A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 &= Q_1 \\ A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 &= Q_2 \\ C\dot{\omega}_3 &= Q_3 \end{aligned}$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

1.4 Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy and Q_i are the non-conservative generalised forces.

1.5 Hamilton's equations

(a) Basic formulation

The generalised momenta p_i and the Hamiltonian $H(\boldsymbol{p},\boldsymbol{q})$ are defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \qquad H(\boldsymbol{p},\boldsymbol{q}) = \sum_i p_i \dot{q}_i - T + V$$

where it should be noted that in the expression for the Hamiltonian the velocities $\dot{q}_i(\boldsymbol{p},\boldsymbol{q})$ must be expressed as a function of the generalised momenta and the generalised displacements.

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i.$$

Special case when the kinetic energy is expressible using a mass matrix $\boldsymbol{M}$:

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1} \boldsymbol{p} \qquad \text{and} \qquad H = T + V$$

(b) Extension topics

The total time derivative of some function $f(\boldsymbol{p},\boldsymbol{q},t)$ can be expressed in terms of the Poisson bracket $\{f, H\}$ in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}, \qquad \{f, H\} \equiv \sum_i \left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right].$$

Common forms of Canonical Transform for Hamilton's equations are:

Type	Generating function	1st eqn	2nd eqn	Kamiltonian
1	$G_1(\boldsymbol{q},\boldsymbol{Q},t)$	$\boldsymbol{p} = \dfrac{\partial G_1}{\partial \boldsymbol{q}}$	$\boldsymbol{P} = -\dfrac{\partial G_1}{\partial \boldsymbol{Q}}$	$K = H + \dfrac{\partial G_1}{\partial t}$
2	$G_2(\boldsymbol{q},\boldsymbol{P},t)$	$\boldsymbol{p} = \dfrac{\partial G_2}{\partial \boldsymbol{q}}$	$\boldsymbol{Q} = \dfrac{\partial G_2}{\partial \boldsymbol{P}}$	$K = H + \dfrac{\partial G_2}{\partial t}$
3	$G_3(\boldsymbol{p},\boldsymbol{Q},t)$	$\boldsymbol{q} = -\dfrac{\partial G_3}{\partial \boldsymbol{p}}$	$\boldsymbol{P} = -\dfrac{\partial G_3}{\partial \boldsymbol{Q}}$	$K = H + \dfrac{\partial G_3}{\partial t}$
4	$G_4(\boldsymbol{p},\boldsymbol{P},t)$	$\boldsymbol{q} = -\dfrac{\partial G_4}{\partial \boldsymbol{p}}$	$\boldsymbol{Q} = \dfrac{\partial G_4}{\partial \boldsymbol{P}}$	$K = H + \dfrac{\partial G_4}{\partial t}$

2 Vibration modes and response

Discrete Systems / **Continuous Systems**

1. Equation of motion

Discrete Systems: The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

Continuous Systems: The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 3 for examples.

2. Kinetic Energy

Discrete Systems:

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

Continuous Systems:

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

Discrete Systems:

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

Continuous Systems: See Section 3 for examples.

4. Natural frequencies and mode shapes

Discrete Systems: The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

Continuous Systems: The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 3) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

Discrete Systems:

$$\mathbf{u}^{(j)^T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)^T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

Continuous Systems:

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

Discrete Systems: The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

Continuous Systems: The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_j q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right] \mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n\zeta_n t} \sin\omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1,x_2,t) = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1,x_2,t) \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n} e^{-\omega_n\zeta_n t} \sin\omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t=0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t=0$ at point x_1, the response at point x_2 is

$$h(x_1,x_2,t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1,x_2,t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

2.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\tilde{T}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 3.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.

3 Governing equations for continuous systems

3.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\big(a^4 - b^4\big)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

3.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$\begin{aligned}
&\cos\alpha\cosh\alpha + 1 = 0, \quad && \alpha_1 = 1.8751\\
&\cos\alpha\cosh\alpha - 1 = 0, \quad && \alpha_1 = 4.7300\\
&\tan\alpha - \tanh\alpha = 0, \quad && \alpha_1 = 3.9266
\end{aligned}$$