EGT2
ENGINEERING TRIPOS PART IIA

Monday 29 April 2024 9.30 to 11.10

Module 3C6

VIBRATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed.

Attachment: 3C5 Dynamics and 3C6 Vibration data sheet 2023 (7 pages).

Engineering Data Book.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) A string of length L has tension P and mass per unit length m. The distance along the string is x, and the small amplitude transverse deflection of the string is denoted $z_s(x,t)$. The string is fixed at $x = 0$ and free at $x = L$.

(i) Find the natural frequencies ω_n and mode shapes $u_n(x)$ for free vibration of the string. [10%]

(ii) Find an expression for transfer function $G_s(x_1, x_2, \omega)$ of the string from an input force applied at $x = x_1$ to an output displacement measured at $x = x_2$, using a modal summation. [10%]

(b) For axial vibration of a uniform fixed-free bar, find an expression for the transfer function $G_b(y_1, y_2, \omega)$ from an input axial force applied at $y = y_1$ to an output axial displacement at $y = y_2$, using a modal summation. The distance along the bar is y and the axial displacement of the bar is $z_b(y,t)$. Take the fixed end of the bar to be at $y = 0$ and assume that the bar has Young's modulus E, density ρ, cross-sectional area A, and length L. [20%]

(c) Under what conditions are the driving point transfer functions of the two systems at their free ends identical, i.e. when is $G_s(L, L, \omega) = G_b(L, L, \omega)$? Write your answer as one or more expressions that relate the properties of the string and the bar. [20%]

(d) The string and bar are coupled as shown in Fig. 1. Assume that the tension in the string does not cause any bending in the bar. For the case when the uncoupled transfer functions are identical, i.e. $G_s(L, L, \omega) = G_b(L, L, \omega)$:

(i) Find the driving point transfer function for the coupled system at the coupling point $(x = y = L)$. [10%]

(ii) Sketch the first four mode shapes of the coupled system and list their natural frequencies. For each mode shape, use a pair of plots to show the string and bar displacements separately. [30%]

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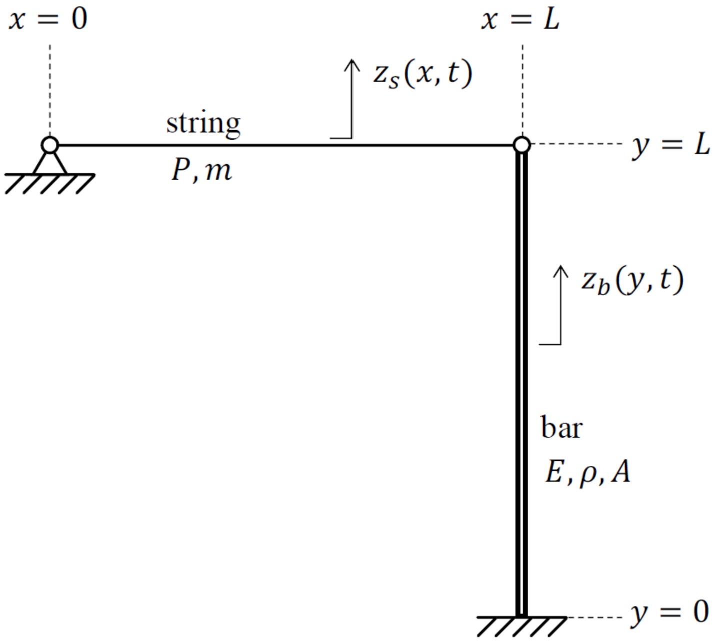

Fig. 1

(TURN OVER

2 A beam of length L, uniform cross-section of area A and second moment of area I is made from a material with density ρ and Young's modulus E. The distance from one end of the beam is x, and the small amplitude transverse deflection of the beam is y. The beam is pinned at both ends.

(a) Starting from the governing equation for transverse vibration of a beam, derive an expression for the natural frequencies ω_n and mode shapes $u_n(x)$ of the beam. [20%]

(b) A mass M is attached to the beam at $x = x_0$ via a light rigid link of length d as shown in Fig. 2. The link and beam are rigidly connected such that they remain at right angles to each other.

(i) Using suitable assumptions, apply Rayleigh's principle to find an approximate expression for the new natural frequencies ω_n' of the modified beam. Write your answer in the form $\omega_n' = \omega_n(1 + \Delta)$ and find Δ. [50%]

(ii) For the case when $M \to \infty, d = 0$ and $x_0 = L/2$, sketch the first four mode shapes and, without further calculation, comment on the natural frequency of each mode. [30%]

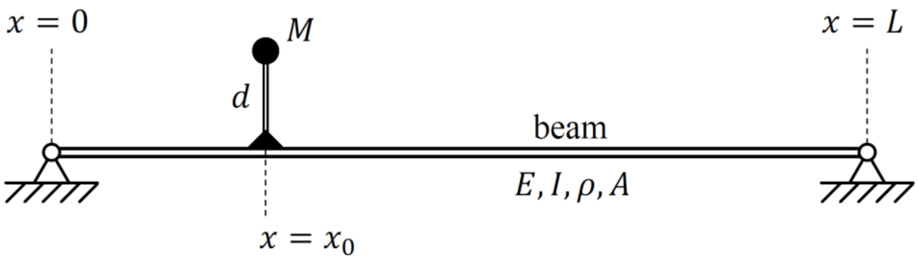

Fig. 2

3 Figure 3 shows a two-dimensional model of a system with three masses *m* that are constrained to move vertically with displacements z_1, z_2 and z_3 and a uniform rigid beam of mass 3*m* and moment of inertia ma^2 about its centre of mass, which can move vertically with displacement *y* and rotate with angle θ. The masses and beam are connected by linear springs of stiffness *k*. The spacing between the masses is *a* and the spacing between the support springs is 2*a*.

(a) Assuming small motions, write down expressions for the kinetic and potential energies of the system *T* and *V* and show that the stiffness matrix can be written:

$$k\begin{bmatrix} 1 & 0 & 0 & -1 & a \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -a \\ -1 & -1 & -1 & 5 & 0 \\ a & 0 & -a & 0 & 4a^2 \end{bmatrix}$$

where the generalized coordinate vector is $[z_1 \quad z_2 \quad z_3 \quad y \quad \theta]^T$. [30%]

(b) The system has two natural modes with mode shapes of the form $[1 \quad 1 \quad 1 \quad \beta \quad 0]^T$, where β is an unknown constant. By substituting this mode shape into the eigenvalue calculation, or by considering a simpler 2-mass system, or otherwise, determine the value of β and determine the two corresponding natural frequencies. [20%]

(c) Three of the mode shapes have $y = 0$. Sketch these modes and put all of the modes in order of increasing natural frequency. [30%]

(d) Assuming that the modes are lightly damped, sketch a log amplitude plot for a transfer function describing the vertical displacement z_1 for a sinusoidal vertical force *f* applied to the mass with coordinate z_3. [20%]

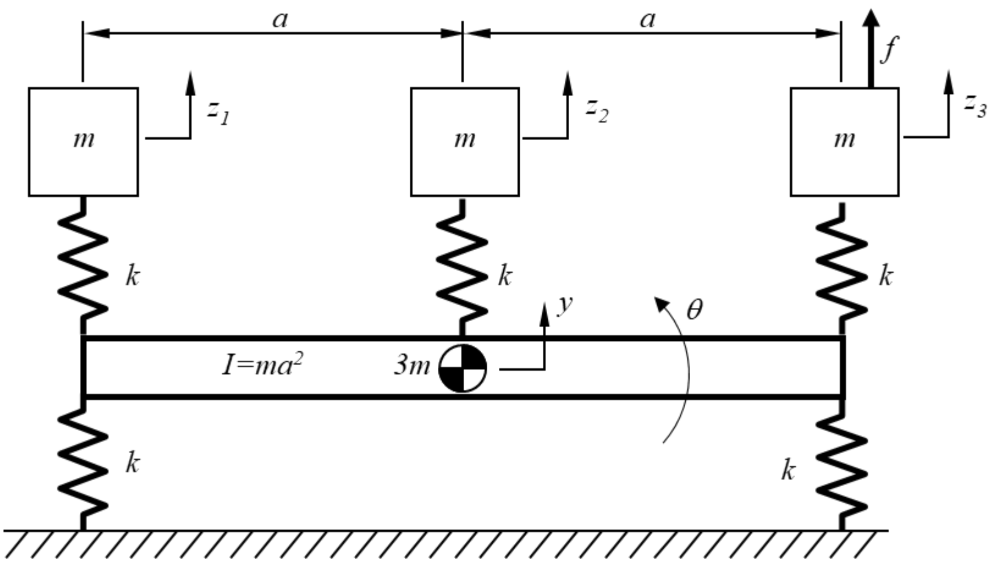

Fig. 3

4 Torsional vibration of an engine is represented by the model shown in Fig. 4. Five discs, each with polar moment of inertia J are connected by shaft sections, three of which have torsional stiffness k and one of which has torsional stiffness S. The assembly is supported by frictionless bearings. The angular positions of the discs are defined by the coordinate vector $[\theta_1 \;\; \theta_2 \;\; \theta_3 \;\; \theta_4 \;\; \theta_5]^{\mathrm{T}}$.

(a) Write expressions for the kinetic and potential energies of the system. [20%]

(b) Using the results from (a), write an expression for Rayleigh's quotient. Show that the quotient gives the correct natural frequency for the rigid body mode of vibration. [20%]

(c) For the case when $S = k$, the mode with the smallest non-zero natural frequency has the form $[1 \quad \alpha \quad 0 \quad -\alpha \quad -1]^{\mathrm{T}}$. By minimising Rayleigh's quotient, find the natural frequency and the corresponding mode shape. Comment on any other modes found by this analysis. [30%]

(d) If the stiffness of shaft S is increased by 10%, (i.e. $S = 1.1k$), estimate the percentage change in the smallest non-zero natural frequency. [30%]

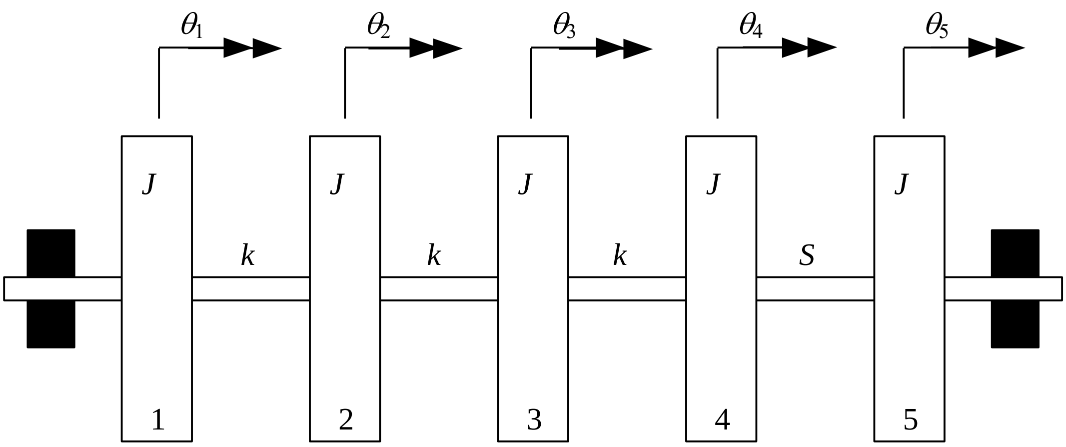

Fig. 4

END OF PAPER

Part IIA Data Sheet

Module 3C5 Dynamics
Module 3C6 Vibration

1 Dynamics in three dimensions

1.1 Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\boldsymbol{v}_{\mathrm{G}}$, M is the total mass, $\boldsymbol{v}_{\mathrm{G}}$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_{\mathrm{G}} - \boldsymbol{r}_{\mathrm{P}}) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_{\mathrm{G}}$$

$$= \dot{\boldsymbol{h}}_{\mathrm{P}} + \dot{\boldsymbol{r}}_{\mathrm{P}} \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here $\boldsymbol{h}_{\mathrm{P}}$ and $\boldsymbol{h}_{\mathrm{G}}$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_{\mathrm{P}}) \times m_i \dot{\boldsymbol{r}}_i$$

$$= \boldsymbol{h}_{\mathrm{G}} + (\boldsymbol{r}_{\mathrm{G}} - \boldsymbol{r}_{\mathrm{P}}) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) dm = \boldsymbol{I}\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $A = \int (y^2 + z^2) dm \qquad B = \int (z^2 + x^2) dm \qquad C = \int (x^2 + y^2) dm$

$D = \int yz\ dm \qquad E = \int zx\ dm \qquad F = \int xy\ dm$

where all integrals are taken over the volume of the body.

1.2 Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1.1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

1.3 Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= Q_1 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= Q_2 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= Q_3 \end{aligned}$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the"Gyroscope equations"):

$$\begin{aligned} A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 &= Q_1 \\ A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 &= Q_2 \\ C\dot{\omega}_3 &= Q_3 \end{aligned}$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

1.4 Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy and Q_i are the non-conservative generalised forces.

1.5 Hamilton's equations

(a) Basic formulation

The generalised momenta p_i and the Hamiltonian $H(\boldsymbol{p},\boldsymbol{q})$ are defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \qquad H(\boldsymbol{p},\boldsymbol{q}) = \sum_i p_i \dot{q}_i - T + V$$

where it should be noted that in the expression for the Hamiltonian the velocities $\dot{q}_i(\boldsymbol{p},\boldsymbol{q})$ must be expressed as a function of the generalised momenta and the generalised displacements.

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i.$$

Special case when the kinetic energy is expressible using a mass matrix $\boldsymbol{M}$:

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1}\boldsymbol{p} \qquad \text{and} \qquad H = T + V$$

(b) Extension topics

The total time derivative of some function $f(\boldsymbol{p},\boldsymbol{q},t)$ can be expressed in terms of the Poisson bracket $\{f,H\}$ in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f,H\}, \qquad \{f,H\} \equiv \sum_i \left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right].$$

Common forms of Canonical Transform for Hamilton's equations are:

Type	Generating function	1st eqn	2nd eqn	Kamiltonian
1	$G_1(\boldsymbol{q},\boldsymbol{Q},t)$	$\boldsymbol{p} = \frac{\partial G_1}{\partial \boldsymbol{q}}$	$\boldsymbol{P} = -\frac{\partial G_1}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_1}{\partial t}$
2	$G_2(\boldsymbol{q},\boldsymbol{P},t)$	$\boldsymbol{p} = \frac{\partial G_2}{\partial \boldsymbol{q}}$	$\boldsymbol{Q} = \frac{\partial G_2}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_2}{\partial t}$
3	$G_3(\boldsymbol{p},\boldsymbol{Q},t)$	$\boldsymbol{q} = -\frac{\partial G_3}{\partial \boldsymbol{p}}$	$\boldsymbol{P} = -\frac{\partial G_3}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_3}{\partial t}$
4	$G_4(\boldsymbol{p},\boldsymbol{P},t)$	$\boldsymbol{q} = -\frac{\partial G_4}{\partial \boldsymbol{p}}$	$\boldsymbol{Q} = \frac{\partial G_4}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_4}{\partial t}$

2 Vibration modes and response

Discrete Systems / **Continuous Systems**

1. Equation of motion

The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 3 for examples.

2. Kinetic Energy

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

See Section 3 for examples.

4. Natural frequencies and mode shapes

The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 3) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

$$\mathbf{u}^{(j)^T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)^T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_{j} q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\text{diag}(\omega_j^2)\right] \mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1,x_2,t) = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1,x_2,t) \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1, the response at point x_2 is

$$h(x_1, x_2, t) = \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

2.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\tilde{\tilde{T}}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 3.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\tilde{\tilde{T}}}$ is *stationary* near each mode.

3 Governing equations for continuous systems

3.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

3.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$
\begin{aligned}
&\cos\alpha\cosh\alpha + 1 = 0, \quad && \alpha_1 = 1.8751 \\
&\cos\alpha\cosh\alpha - 1 = 0, \quad && \alpha_1 = 4.7300 \\
&\tan\alpha - \tanh\alpha = 0, \quad && \alpha_1 = 3.9266
\end{aligned}
$$