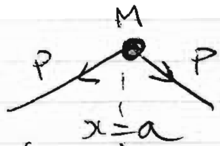
3C6: Vibration Solutions 2014

1 (a) Near the mass M:

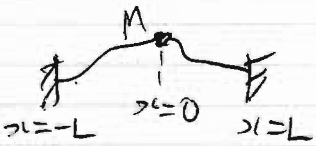


Newton II $\rightarrow$

$$M\frac{\partial^2 w}{\partial t^2}(a,t) = P\left[\frac{\partial w}{\partial x}(a+,t) - \frac{\partial w}{\partial x}(a-,t)\right]$$

Also need $w(a-,t) = w(a+,t)$

(b) Governing equation $P\dfrac{\partial^2 w}{\partial x^2} = m\dfrac{\partial^2 w}{\partial t^2}$



For a mode, let $w = u(x)e^{i\omega t}$

Then $Pu'' = -m\omega^2 u$, with general solution
$u = A\cos kx + B\sin kx$, $k^2 = \dfrac{m\omega^2}{P}$, A, B constants.

Solution has to look like this both sides of the mass M.
So let
$$u = \begin{cases} K_1 \sin k(x+L) & -L \le x \le 0 \\ K_2 \sin k(L-x) & 0 \le x \le L \end{cases}$$
to satisfy $u=0$ at $x=-L$, $x=L$.

For symmetric modes, $K_1 = K_2$, for antisymmetric modes $K_1 = -K_2$

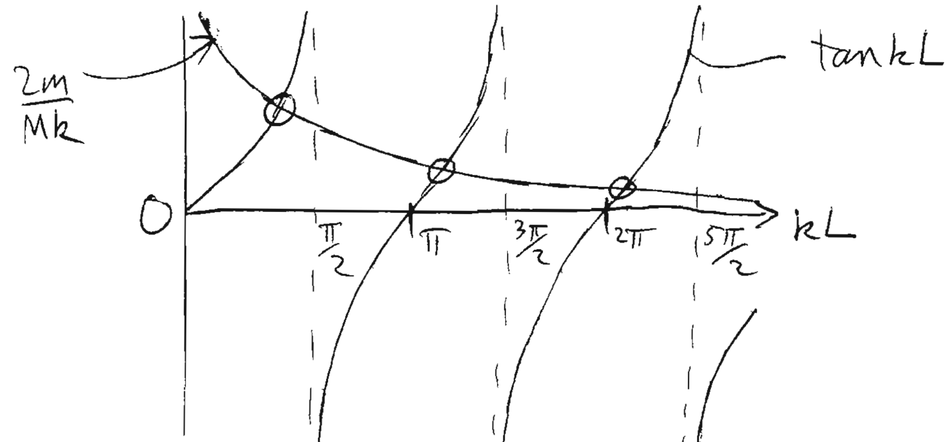
Symmetric case: result from part (a) requires at $x=0$
$$-M\omega^2 \sin kL = 2P(-k\cos kL)$$
$$\therefore \tan kL = \frac{2Pk}{M\omega^2} = \frac{2P}{M}\frac{k}{(Pk^2/m)} = \frac{2m}{Mk}$$

Solutions for k then give $\omega = k\sqrt{\dfrac{P}{m}}$

Antisymmetric case: must have $u(0)=0$
$\therefore \sin kL = 0$, so $kL = n\pi$, $n = 1,2,3\ldots$

Then $\omega = \dfrac{n\pi}{L}\sqrt{\dfrac{P}{m}}$, $n=1,2,3\ldots$

1(c) Plot $\tan kL$, $\frac{2m}{Mk}$ versus k and see where they cross.



There is always one root between $kL = n\pi$ and $(n+\frac{1}{2})\pi$, for $n = 0, 1, 2, 3 \ldots$

Note that the points $n\pi$ are the antisymmetric mode roots.

For $M \to 0$, the hyperbola moves up in the diagram so the roots tend to $(n+\frac{1}{2})\pi$, as expected for an unweighted string of length $2L$.

For $M \to \infty$, roots tend to $n\pi$. The mass doesn't move, dividing the string into equal pieces, so these are matching frequencies in each half

(d) For M very small can assume the mode shapes don't change significantly. So use Rayleigh with the added mass but the mode shape for no mass.

Lowest mode is $u = \cos\frac{\pi x}{2L}$

So from Data Sheet
$$\omega^2 \simeq \frac{\frac{1}{2} P \int_{-L}^{L} u'^2 \, dx}{\frac{1}{2} m \int_{-L}^{L} u^2 \, dx + \frac{1}{2} M u^2(0)}$$

$$= \frac{P\left(\frac{\pi}{2L}\right)^2 \cdot L}{m \cdot L + M} \quad \text{since } \int_{-L}^{L} \cos^2 \frac{\pi x}{2L} \, dx = L$$

2 (a) Equation: $EI\frac{\partial^4 w}{\partial x^4} - \rho A \frac{\partial^2 w}{\partial t^2} = 0$

For a mode $w = u(x)e^{i\omega t}$

$\therefore \frac{d^4 u}{dx^4} = \alpha^4 u, \quad \alpha^4 = \frac{\rho A}{EI}\omega^2$ (1)

General solution $u = K_1 \cos\alpha x + K_2 \sin\alpha x + K_3 \cosh\alpha x + K_4 \sinh\alpha x$

At $x=0$: $\begin{cases} u=0 \rightarrow K_1 + K_3 = 0 \\ u''=0 \rightarrow \alpha^2(-K_1 + K_3) = 0 \end{cases}$

So $K_1 = K_3 = 0$

At $x=L$: $\begin{cases} u''=0 \rightarrow \alpha^2(-K_2 \sin\alpha L + K_4 \sinh\alpha L) = 0 \\ u'''=0 \rightarrow \alpha^3(-K_2 \cos\alpha L + K_4 \cosh\alpha L = 0 \end{cases}$

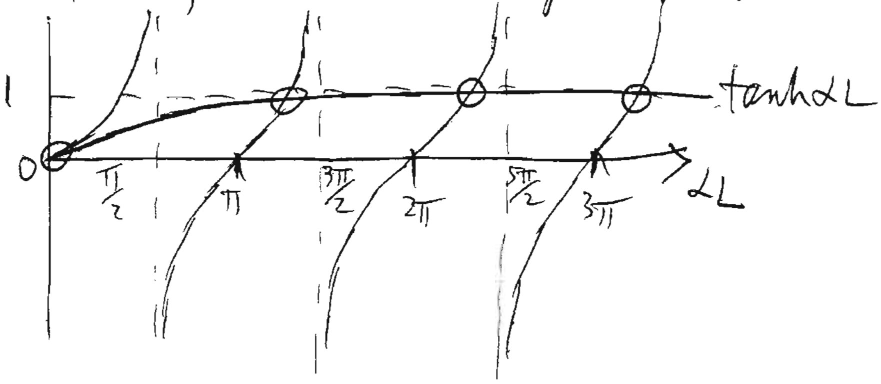
For non-trivial solution need $\begin{vmatrix} -\sin\alpha L & \sinh\alpha L \\ -\cos\alpha L & \cosh\alpha L \end{vmatrix} = 0$

$\therefore \sin\alpha L \cosh\alpha L = \cos\alpha L \sinh\alpha L$

$\therefore \tan\alpha L = \tanh\alpha L$

Roots for α give ω from (1).

(b) Plot $\tan\alpha L$, $\tanh\alpha L$ & look for intersection



One root at $k=0 \rightarrow \omega = 0$: rigid-body rotation about $x=0$.

Other roots close to where $\tan\alpha L = 1$

i.e. $\alpha L = \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4} \ldots (n+\frac{1}{4})\pi$

Then $\omega \simeq \sqrt{\frac{EI}{\rho A}}\left(\frac{(n+1/4)\pi}{L}\right)^2$ where $\begin{cases} I = \frac{1}{12}bh^3 \\ A = bh \end{cases}$

2(b) contd. Mode shapes

(1)  rigid rotation

(2)  one nodal point

(3)  two nodal points

(c) One semitone is a ratio of $\sqrt[12]{2} = \lambda$ say

ie if f_2 is z semitones above f_1,

then $\frac{f_2}{f_1} = \lambda^z$ ie $z = \frac{\log(f_2/f_1)}{\log \lambda}$ ($\log_{10}$ or ln)

$\frac{f_2}{f_1} \triangleq \left(\frac{9}{5}\right)^2 \rightarrow z = 20.35$ semitones

$\frac{f_3}{f_1} \triangleq \left(\frac{13}{5}\right)^2 \rightarrow z = 33.08$ semitones

(d) Gravity has two effects. First, the rigid-body mode becomes a pendulum vibration, and it now has a non-zero frequency given by the usual pendulum calculation.

Second the self-weight of the hanging beam creates a tensile stress in the beam which varies with x, from zero at the bottom (free) end to a maximum at the top.

This introduces an additional restoring force as in a stretched string, which must be added to the bending term we already have. It is more complicated than the simple expression for a string because the tension is a function of x.

3 (a) $T = \frac{1}{2}\left[m\dot{y}_1^2 + 2m\dot{y}_2^2 + 3m\dot{y}_3^2 + 4m\dot{y}_4^2\right]$

or $\tilde{T} = \frac{1}{2}m\left[y_1^2 + 2y_2^2 + 3y_3^2 + 4y_4^2\right]$

$V = \frac{1}{2}\left[k(y_1-y_2)^2 + 2k(y_3-y_2)^2 + 3k(y_4-y_3)^2 + 4ky_4^2\right]$

(b)(i) Load F on top mass: all spring forces must be equal to F for equilibrium at each mass.

So $\begin{cases} k(y_2-y_1) = -F, & 2k(y_3-y_2) = -F \\ 3k(y_4-y_3) = -F, & -4ky_4 = -F \end{cases}$

Choose $F = k$:

Then $y_4 = 1/4$, $y_3 = y_4 + 1/3 = 7/12$,

$y_2 = y_3 + 1/2 = 13/12$, $y_1 = y_2 + 1 = 25/12$

So choose $[y_1, y_2, y_3, y_4] = [25, 13, 7, 3]$

(ii) Self-weight: each spring force must balance the total weight above it

$\therefore \begin{cases} k(y_1-y_2) = mg \\ 2k(y_2-y_3) = 3mg \\ 3k(y_3-y_4) = 6mg \\ 4k(y_4) = 10mg \end{cases}$

Choose $\frac{mg}{k} = 1$ to get ratios.

Then $y_4 = \frac{5}{2}$, $y_3 = y_4 + 2 = \frac{9}{2}$,

$y_2 = y_3 + \frac{3}{2} = \frac{12}{2}$, $y_1 = y_2 + 1 = \frac{14}{2}$

So can choose ratios $[y_1, y_2, y_3, y_4] = [14, 12, 9, 5]$.

3(c) Now use in Rayleigh:

(i) $$\omega^2 \simeq \frac{\frac{1}{2}k\left[12^2 + 2.6^2 + 3.4^2 + 4.3^2\right]}{\frac{1}{2}m\left[25^2 + 2.13^2 + 3.7^2 + 4.3^2\right]} = \frac{300k}{1146m}$$

$$\therefore \omega^2 \simeq 0.2618\frac{k}{m} \rightarrow \omega = 0.5117\sqrt{\frac{k}{m}}$$

(ii) $$\omega^2 \simeq \frac{\frac{1}{2}k\left[2^2 + 2.3^2 + 3.4^2 + 4.5^2\right]}{\frac{1}{2}m\left[14^2 + 2.12^2 + 3.9^2 + 4.5^2\right]} = \frac{170k}{827m}$$

$$\therefore \omega^2 \simeq 0.2056\frac{k}{m} \rightarrow \omega = 0.4534\sqrt{\frac{k}{m}}$$

The second approximation is smaller so gives the more accurate estimate of the lowest frequency.

An inkling of the reason is that each "separate oscillator" in the column has the same ratio of k/m, so has the same "local resonance frequency".

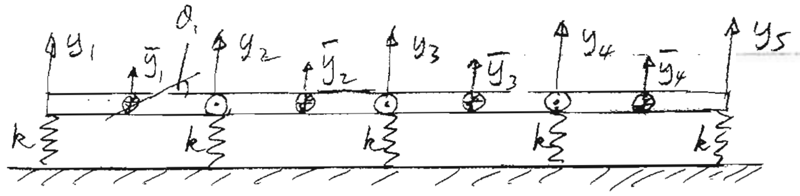
So perhaps we should expect roughly equal extension in all four springs? If we try the estimate $[y_1, y_2, y_3, y_4] = [4, 3, 2, 1]$

we obtain $$\omega^2 \simeq \frac{\frac{1}{2}k\left[1 + 2 + 3 + 4\right]}{\frac{1}{2}m\left[4^2 + 2.3^2 + 3.2^2 + 4\right]} = 0.2\frac{k}{m}$$

which is indeed a lower frequency and thus a still better estimate.

[NB from Matlab, exact frequency is $0.1988\frac{k}{m} = \omega^2$]

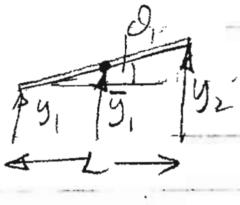
4



(a) PE: $V = \frac{1}{2}k\left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right)$

KE: $T = \frac{1}{2}m\left(\dot{\bar y}_1^2 + \dot{\bar y}_2^2 + \dot{\bar y}_3^2 + \dot{\bar y}_4^2\right) + \frac{1}{2}I\left(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2 + \dot\theta_4^2\right)$

with $I = \dfrac{mL^2}{12}$



$\bar y_1 = \dfrac{y_1 + y_2}{2}$

$\theta_1 = \dfrac{y_2 - y_1}{L}$

Consider first mass only:

$$T_1 = \frac{1}{2}m\left(\frac{\dot y_1 + \dot y_2}{2}\right)^2 + \frac{1}{2}\,\frac{m\cancel{L^2}}{12}\left(\frac{y_2 - y_1}{\cancel{L}}\right)^2$$

$$= \frac{1}{2}m\left[\frac{\dot y_1^2}{4} + \frac{\dot y_2^2}{4} + \frac{\cancel{2}\dot y_1 \dot y_2}{\cancel{4}_2} + \frac{\dot y_2^2}{12} + \frac{\dot y_1^2}{12} - \frac{\cancel{2}\dot y_1 \dot y_2}{\cancel{12}_6}\right]$$

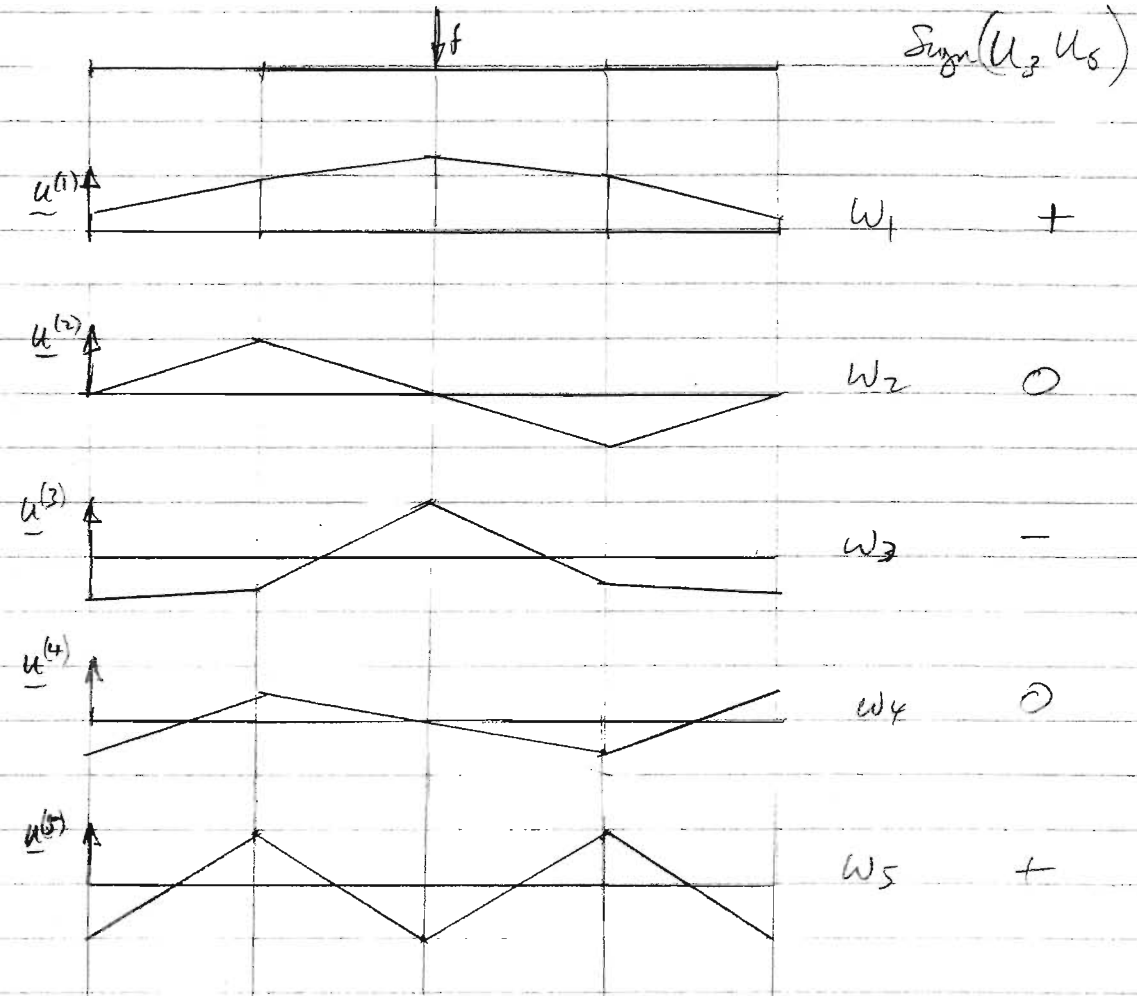
$$= \frac{1}{2}m\left[\frac{\dot y_1^2}{3} + \frac{\dot y_2^2}{3} + \frac{\dot y_1 \dot y_2}{3}\right]$$

So overall: $T = \frac{1}{2}\frac{m}{3}\left[\dot y_1^2 + \dot y_2^2 + \dot y_1 \dot y_2 + \dot y_2^2 + \dot y_3^2 + \dot y_2 \dot y_3 + \text{---}\right]$

$$= \frac{1}{2}\frac{m}{3}\left[\dot y_1^2 + 2\dot y_2^2 + 2\dot y_3^2 + 2\dot y_4^2 + \dot y_5^2 + \dot y_1\dot y_2 + \dot y_2\dot y_3 + \dot y_3\dot y_4 + \dot y_4\dot y_5\right.$$

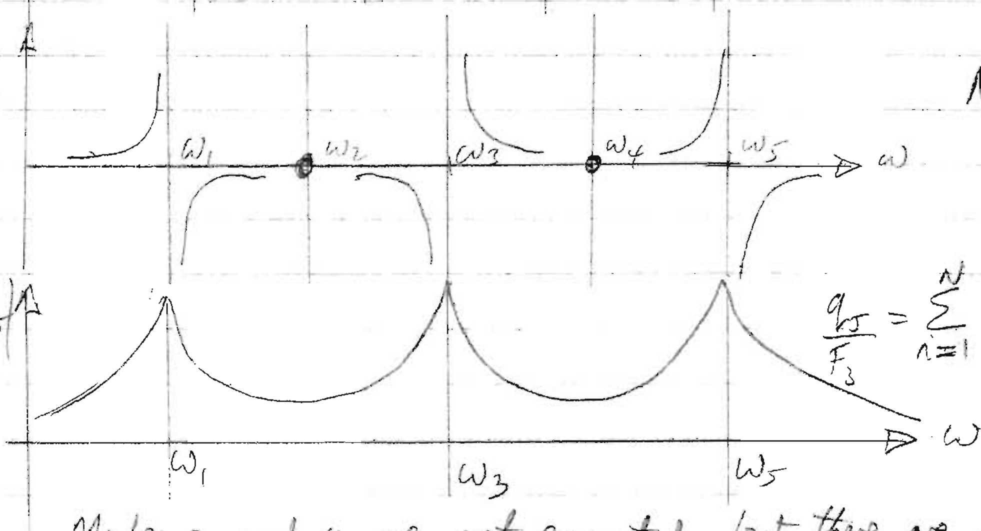
Giving $[M] = m\begin{bmatrix} 1/3 & 1/6 & & & \\ 1/6 & 2/3 & 1/6 & & \\ & 1/6 & 2/3 & 1/6 & \\ & & 1/6 & 2/3 & 1/6 \\ & & & 1/6 & 1/3 \end{bmatrix}\begin{array}{c}\dot y_1\\ \dot y_2\\ \dot y_3\\ \dot y_4\\ \dot y_5\end{array}$ & $K = \begin{bmatrix} k & & & & \\ & k & & & \\ & & k & & \\ & & & k & \\ & & & & k \end{bmatrix}$

4(b) The mode shapes are all symmetric or anti-symmetric with 0, 1, 2, 3, 4 zero crossings:



(c) H_{35}

No anti-resonances



$$\frac{q_5}{F_3} = \sum_{n=1}^{N} \frac{U_3^{(n)} u_5^{(n)}}{\omega_n^2 - \omega^2}$$

Modes 2 and 4 are not excited but there are minima at ω_2 & ω_4 due to superposition of the adjacent modes.

4 (d) For antisymmetric motion, the vector of displacements must take the form $\begin{bmatrix} a & b & 0 & -b & -a \end{bmatrix}^t$

for some values a and b.

Then $T = \frac{m}{3}\left[\dot{a}^2 + \dot{b}^2 + \dot{a}\dot{b} + \dot{b}^2\right]$

so the new 2×2 mass matrix is $M' = \frac{m}{3}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

Similarly, $V = ka^2 + kb^2$

So new stiffness matrix is $K' = 2k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Writing $\Omega^2 = \frac{m}{6k}\omega^2$, the natural frequencies satisfy

$$\begin{vmatrix} 1-2\Omega^2 & -\Omega^2 \\ -\Omega^2 & 1-4\Omega^2 \end{vmatrix} = 0$$

$$\therefore (1-2\Omega^2)(1-4\Omega^2) - \Omega^4 = 0$$

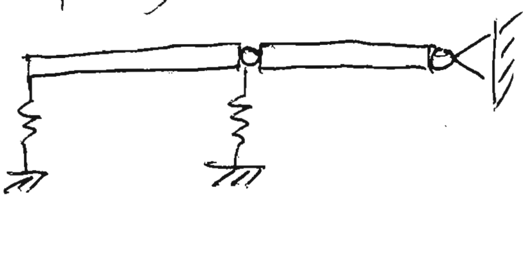
$$\therefore 7\Omega^4 - 6\Omega^2 + 1 = 0$$

$$\therefore \Omega^2 = \frac{1}{14}\left(6 \pm \sqrt{36-28}\right) = \frac{1}{7}\left(3 \pm \sqrt{2}\right)$$

$$\therefore \omega = \sqrt{\frac{6k}{7m}\left(3 \pm \sqrt{2}\right)} = \sqrt{\frac{k}{m}}\begin{cases} 1.166 \\ 1.945 \end{cases}$$

Physical view of antisymmetric modes: $y_3 = 0$, so this point acts like a fixed pivot.

Consider one half only, and hence see 2 DoF:



Q1 Mass-loaded string

The least popular question, but well done by many. Part (b) caused the biggest difficulties, many failing to formulate the problem clearly so that symmetric and antisymmetric modes could be analysed separately. Physical insight to interpret parts (c) and (d) was generally good.

Q2 Pinned-free beam

Most did part (a) well, although a few solved the clamped, rather than the pinned, problem. For part (d), surprisingly few realised that the rigid-body mode becomes a pendulum mode at non-zero frequency. A larger number thought about the effect of tension due to self-weight.

Q3 Four-DOF Rayleigh calculation

Most understood how to apply and interpret Rayleigh's principle, but many did not succeed in correctly working out the two static calculations for part (b).

Q4 Four-DOF modal calculations and transfer function

Disappointingly many candidates failed to include the rotational kinetic energy in part (a). A few did the converse: including rotational energy but missing the translational terms. The qualitative sketching of modes for part (b) was very revealing about depth of physical insight. Part (d) was rather well done, many candidates understanding clearly how to deal with the restricted problem of antisymmetric modes.

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