EGT2
ENGINEERING TRIPOS PART IIA

Friday 30 April 2021 9 to 10.40

Module 3C6

VIBRATION

Solutions

Question 1

1 A cantilever beam of length L, uniform cross-section of area A and second moment of area I is made of a material with density ρ and Young's modulus E.

(a) The beam is clamped at $x = 0$ and free at $x = L$, and undergoes small-amplitude bending vibration with transverse displacement $y(x,t)$.

(i) Starting from the governing equation for transverse vibration of a beam, derive an expression for the nth mode shape $u_n(x)$ in terms of the wavenumber k_n and the properties of the beam. [20%]

$$EIy^{iv} + \rho A \ddot{y} = 0.$$

let $y = u(x) e^{i\omega t}$

$$\Rightarrow \quad EIu^{iv} - \rho A \omega^2 u = 0.$$

giving $u = A\cos kx + B\sin kx + C\cosh kx + D\sinh kx$

$$u' = k(-A\sin kx + B\cos kx + C\sinh kx + D\cosh kx)$$

$$u'' = k^2(-A\cos kx - B\sin kx + C\cosh kx + D\sinh kx)$$

$$u''' = k^3(A\sin kx - B\cos kx + C\sinh kx + D\cosh kx)$$

BC's: @ $x = 0$, $y = 0 \Rightarrow A + C = 0.$

$$A = -C$$

$$y' = 0 \Rightarrow B + D = 0$$

$$B = -D.$$

@ $x = L$, $y'' = 0 \Rightarrow -A\cos kL - B\sin kL - A\cosh kL - B\sinh kL = 0$ — (1)

$y''' = 0 \Rightarrow A\sin kL - B\cos kL - A\sinh kL - B\cosh kL = 0.$ — (2)

wait mode shape not k_n, so need to eliminate B.

$$① \Rightarrow A(\cos kL + \cosh kL) + B(\sin kL + \sinh kL) = 0.$$

giving: $$B = -\frac{A(\cos kL + \cosh kL)}{(\sin kL + \sinh kL)} \equiv -AR \quad (\text{defining } R)$$

So: $$u_n(x) = A(\cos k_n x - R\sin k_n x - \cosh k_n x + R\sinh k_n x)$$

note:
$-AR = B$
$AR = -B.$

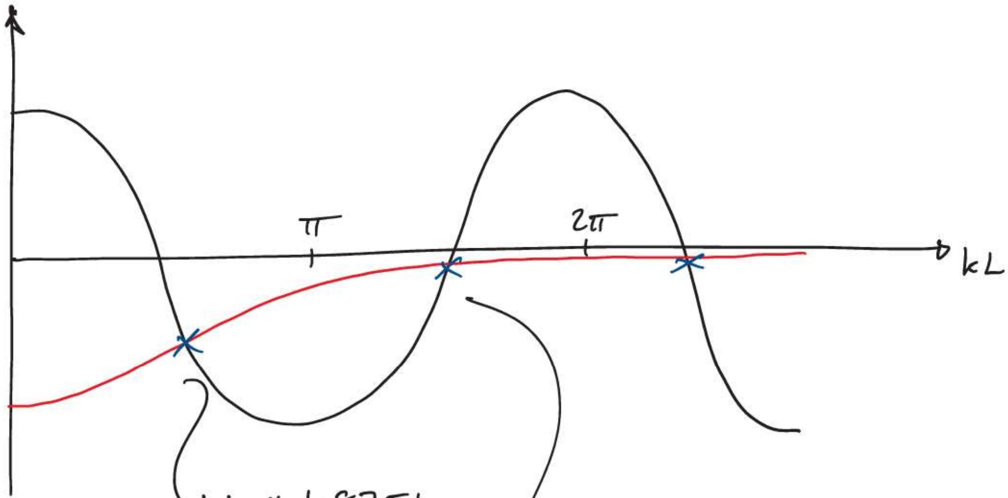
where the ratio $$R = \frac{\cos k_n L + \cosh k_n L}{\sin k_n L + \sinh k_n L}$$

It appears that the shear B.C. giving eq ② doesn't affect the mode shape: but it does because it is needed to derive k_n and the mode shape is a function of k_n.

(ii) Find the natural frequencies ω_n of the beam for the first four modes, using a suitable approximation where needed. Express your answer as factors α_n of the first natural frequency ω_1, i.e. in the form $\omega_n = \alpha_n\omega_1$. [10%]

Clamped-free beam natural frequencies satisfy:

$$\cos kL \cosh kL = -1$$



$k_1 L \simeq 1.8751$

$k_n L \simeq \left(n-\frac{1}{2}\right)\pi$ good approximation for $n = 2, 3, \ldots$

$$\omega_n = k_n^2\sqrt{\frac{EI}{\rho A}} \quad \text{so} \quad \omega_n \propto k_n^2$$

n (mode)	$k_n L$	$(k_n L)^2$	$\alpha_n = \dfrac{(k_n L)^2}{3.516}$
1	1.8751	3.516	1
2	4.712	22.21	6.316
3	7.854	61.69	17.54
4	10.996	120.9	34.39

(b) A single degree-of-freedom mass-on-spring is now connected to the beam at $x = L$, with mass m and stiffness k, as shown in Fig. 1. The beam-spring connection is a frictionless pin, and the mass displacement is indicated by $z(t)$. The natural frequency of the uncoupled mass-on-spring is defined as $\Omega = \sqrt{k/m}$.

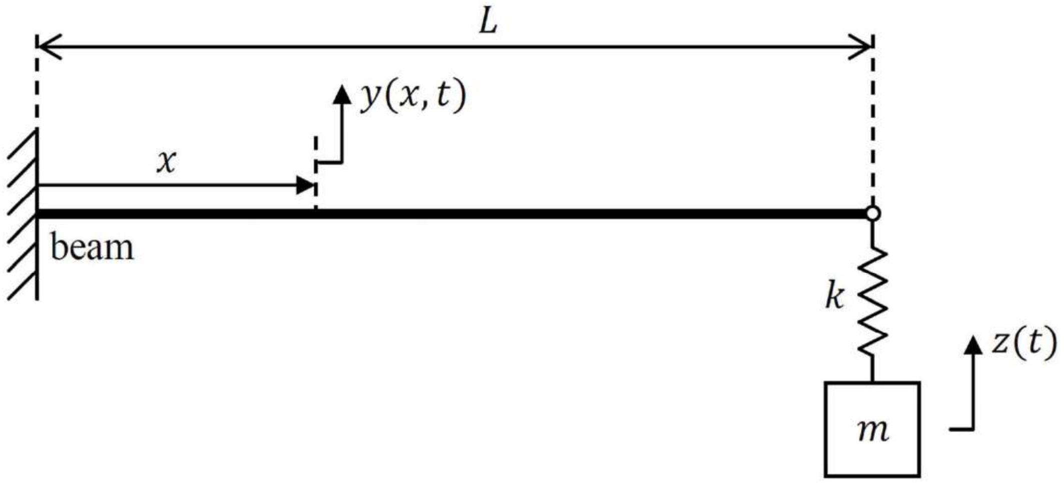


Fig. 1

(i) For the case $k \to 0$, what are the first four natural frequencies of the coupled system? [10%]

$k \to 0$ implies no connection:

$\omega_1' = 0$ (rigid body mode of free mass)

$\omega_2' = \omega_1$

$\omega_3' = \omega_2$

$\omega_4' = \omega_3$

first three modes of beam, unaffected by mass.

(ii) For the case $k \to \infty$, qualitatively describe what you expect for the first four natural frequencies of the coupled system. [10%]

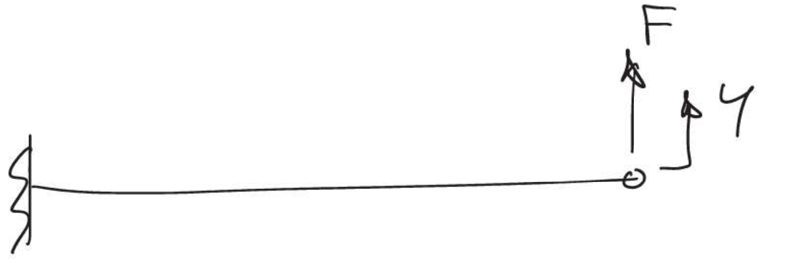
As $k \to \infty$ implies rigid connection.
Expect effect to be same as adding a point mass, so natural frequencies will be slightly smaller than original beam modes. The decrease in frequency will be larger for the higher modes but always limited by

the antiresonances of the driving point transfer function at the free end of the original beam.

(iii) Derive an expression whose solutions give the natural frequencies of the coupled system. Write your answer in terms of the driving point transfer function $H(L, L, \omega)$ of the uncoupled beam and any other system parameters that are needed.

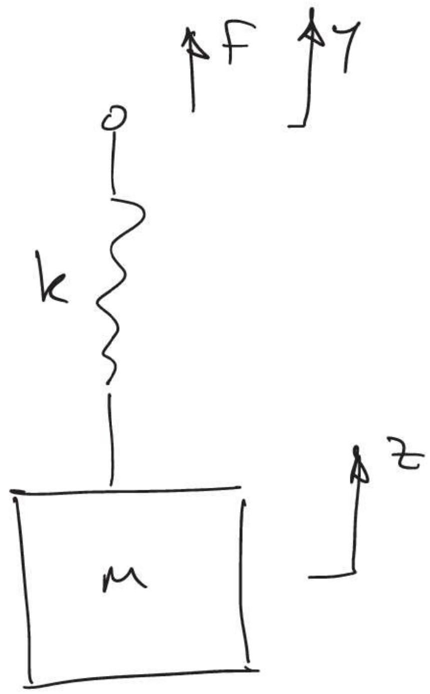
[30%]

Need driving point transfer functions of uncoupled systems at connection:



$$\frac{Y}{F} = H(L, L, \omega)$$

Care is needed when considering the mass on spring:



TF needed is Y/F so need to eliminate z.

$$F = k(Y - z)$$

$$\& \quad F = -m\omega^2 z \qquad (\text{from } f = m\ddot{z}).$$

$$\Rightarrow z = -F/m\omega^2$$

$$\text{so} \quad F = k\left(y + F/m\omega^2\right)$$

$$F\left(1 - k/m\omega^2\right) = ky$$

$$\frac{y}{F} \equiv G = \frac{1}{k} - \frac{1}{m\omega^2} \quad //.$$

now apply coupling formula:

$$G_T = \left(\frac{1}{H} + \frac{1}{G}\right)^{-1} = \frac{GH}{G+H}.$$

new natural frequencies occur when $H = -G$

$$\text{ie} \quad \frac{1}{m\omega^2} - \frac{1}{k} = H(L, L, \omega) \quad //$$

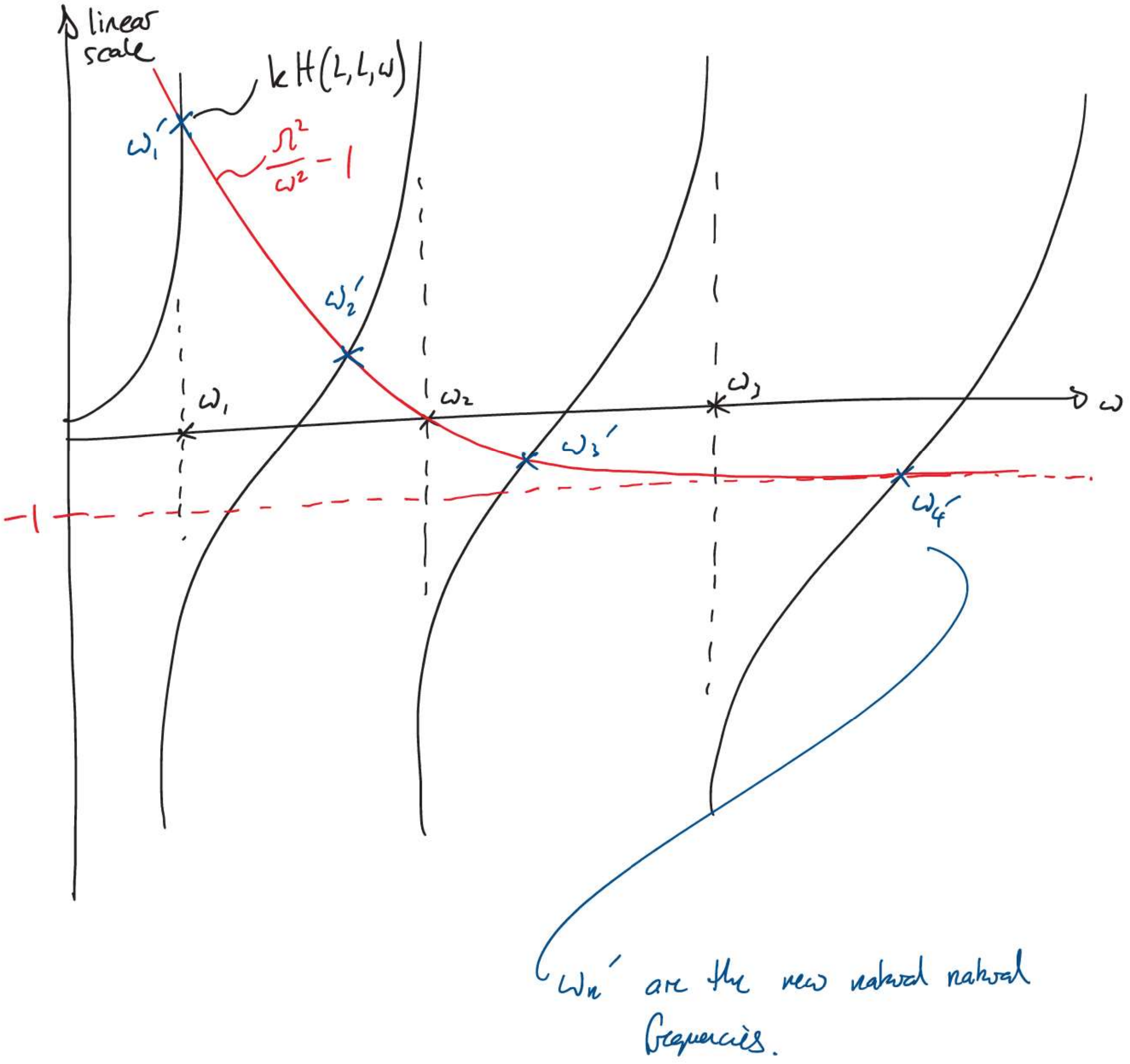
$$\times k \Rightarrow \quad \frac{\Omega^2}{\omega^2} - 1 = k\,H(L, L, \omega) \quad //$$

(any valid form accepted).

(iv) For the case when $\Omega \approx \omega_2$ (i.e. when the natural frequency of the mass-on-spring is tuned to be close to the second natural frequency of the uncoupled beam), use a graphical construction to show the effect of the mass-on-spring on the natural frequencies of the coupled system ω_n'. [20%]

When $\Omega = \omega_2 \Rightarrow \sqrt{k/m} = \Omega = \omega_2$.

note when $\omega = \Omega$ then LHS of condition above is zero.



Note that when $\Omega = \omega_2$, the effect is to 'split' the second beam mode into a pair of new frequencies. This is what happens when a tuned mass absorber is added to a structure.

Question 2

2 A straight pipe is to be inspected for a possible crack. It is proposed that an axial vibration test be carried out to look for changes in the natural frequencies. The pipe has length L and cross-sectional area A. It is made from a material of density ρ and Young's modulus E. The pipe can undergo small amplitude axial oscillations $y(x,t)$ where x is the distance from one end of the pipe and t is time. The ends of the pipe are free.

(a) For an undamaged pipe:

(i) find expressions for the mode shapes and natural frequencies of the pipe. [10%]

<u>Full derivation:</u>

$$\rho A \ddot{y} - EA y'' = 0.$$

let $y = U(x) e^{i\omega t}$

$$\Rightarrow +\rho A \omega^2 U + EA U'' = 0.$$

let $U = e^{ikx}$

$$\Rightarrow \rho A \omega^2 - EA k^2 = 0 \quad \Rightarrow \omega = k\sqrt{E/\rho}$$

General solution: $U = D_1 \cos kx + D_2 \sin kx$.

$U' = -D_1 k \sin kx + D_2 k \cos kx$.

BC's: $U'(0) = 0 \Rightarrow D_2 = 0$.

$U'(L) = 0 \Rightarrow -D_1 \sin kL = 0$.

$\sin kL = 0$

$kL = n\pi$

$k_n = \frac{n\pi}{L}$

natural frequencies: $\omega_n = \frac{n\pi}{L}\sqrt{E/\rho}$

mode shapes: $U_n = \cos \frac{n\pi x}{L}$

$n = 0, 1, 2 \ldots$

rigid body mode

Solution without derivation also accepted.

(ii) find a summation expression for the driving point transfer function $G(x, x, \omega)$ from input force to output axial displacement at a distance x from one end of the pipe, allowing for light damping. [20%]

Mass normalise $u_n(x)$:

$$\int u_n^2(x)\, dm = 1$$

$$\int_0^L D_1^2 \cos^2 \frac{n\pi x}{L} \cdot \rho A\, dx = 1 .$$

$$\rho A D_1^2 \int_0^L \frac{1}{2}\left(1 + \cos \frac{2n\pi x}{L}\right) dx = 1$$

$$\frac{\rho A D_1^2}{2} L = 1 \quad \rightarrow \quad D_1^2 = \frac{2}{\rho A L}$$

then $$G(x,x,\omega) = \frac{2}{\rho A L} \sum_{n=0}^{\infty} \frac{\cos^2\left(\frac{n\pi x}{L}\right)}{\omega_n^2 + 2j\zeta\omega_n\omega - \omega^2}$$ //

(b) A small crack at a distance $x = a$ from one end of the pipe can be modelled as a local reduction in axial stiffness. The potential energy of the cracked pipe can be written:

$$V = V_0 - \Delta V$$

where V_0 is the potential energy of an undamaged pipe and ΔV is the reduction in potential energy caused by the crack which can be written:

$$\Delta V = \frac{1}{2} C \left(\left. \frac{\partial y}{\partial x} \right|_{x=a} \right)^2$$

C is a constant that corresponds to the reduction in stiffness and the term in brackets indicates the first derivative of displacement evaluated at $x = a$. All other properties of the pipe can be assumed to be unchanged from part (a).

(i) Use Rayleigh's principle to find an approximate expression for the new natural frequency ω_n' of the nth mode of the pipe for a given stiffness reduction C. [30%]

$$\omega_n'^2 \simeq \frac{V}{\tilde{T}} = \omega_n^2 - \frac{\Delta V}{\tilde{T}}$$

$$= \omega_n^2 - \frac{\frac{1}{2} C \left. u' \right|_a^2}{\frac{1}{2}\rho A \int_0^L u^2\, dx}$$

need: $u = \cos\frac{n\pi x}{L}$

$$u' = -\frac{n\pi}{L}\sin\frac{n\pi x}{L}, \quad \text{so} \quad u'\big|_a = -\frac{n\pi}{L}\sin\frac{n\pi a}{L}$$

& : $$\int_0^L u^2\,dx = \int_0^L \cos^2\frac{n\pi x}{L}\,dx = \int_0^L \frac{1}{2}\left(1+\cos\frac{2n\pi x}{L}\right)dx = \frac{L}{2}$$

hence $$\omega_n'^2 \simeq \omega_n^2 - \frac{C\left(\frac{n\pi}{L}\right)^2\left(\sin\frac{n\pi a}{L}\right)^2}{\rho A\,L/2}$$

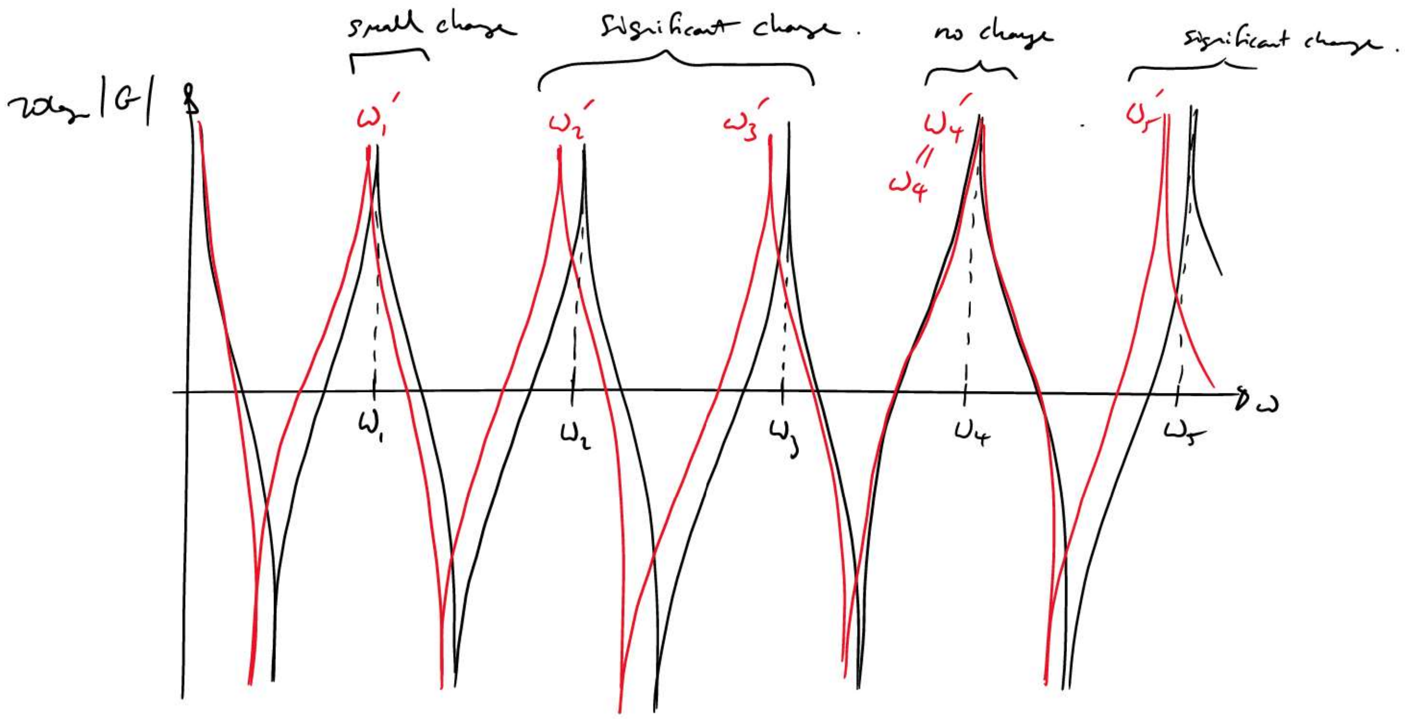
$$\omega_n'^2 \simeq \omega_n^2 - \frac{2C\left(n\pi\sin\frac{n\pi a}{L}\right)^2}{\rho A L^3}$$

(ii) Sketch the driving point transfer function $G(L,L,\omega)$ for the undamaged pipe of part (a) and when there is a small crack at a distance $x = L/4$ from one end. Highlight which modes are most and least affected by the presence of the crack. [30%]

Note: crack affects frequencies most when gradient of mode shape greatest: no effect on $n = 0, 4, 8\ldots$ & increasing effect with increasing frequency.

Change in frequency depends on $n\sin\frac{n\pi a}{L}$, so at $a = L/4$:

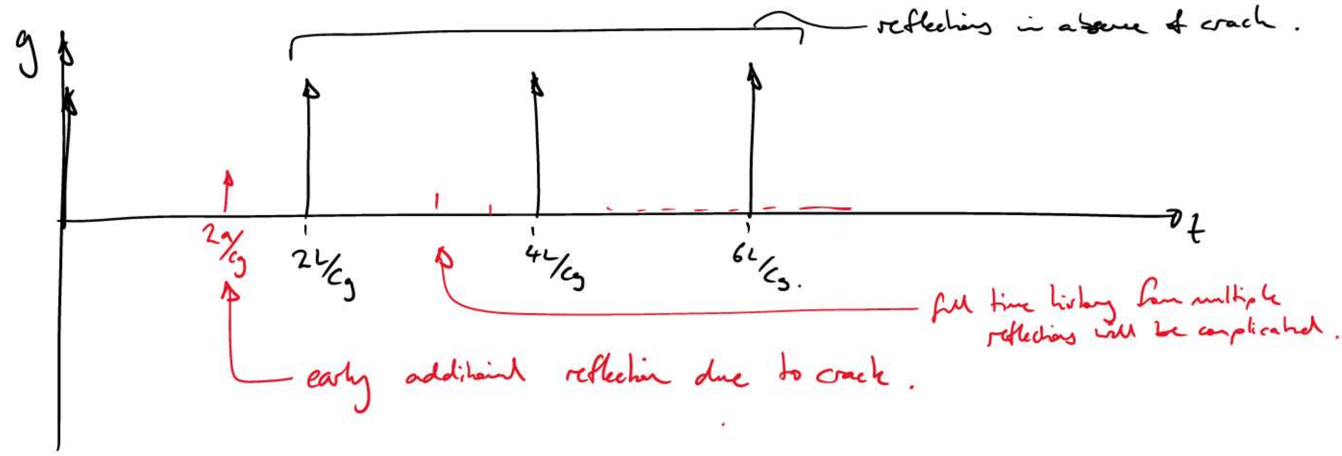
n	$\left\lvert n\sin\frac{n\pi}{4}\right\rvert$	effect of crack
0	0	no effect
1	0.7	relatively small
2	2	significant
3	2.1	significant (but fractionally smaller than $n=2$)
4	0	no effect
5	3.5	significant



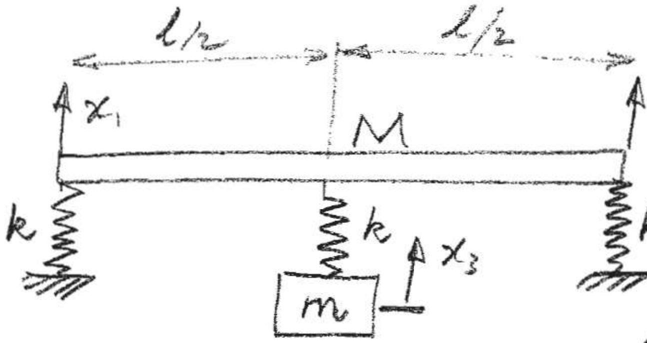
(iii) Briefly describe how the crack could be detected using a wave-based approach.

[10%]

By applying an impulse (or other transient) at an end of the pipe, measure the transient response. A crack would cause reflections, & the distance to the crack could be determined from the group velocity for axial wave propagation.



3. (a)



KE: $T = \frac{1}{2} m \dot{x}_3^2 + \frac{1}{2} M \left(\frac{\dot{x}_1 + \dot{x}_2}{2} \right)^2 + \frac{1}{2} I \left(\frac{\dot{x}_2 - \dot{x}_1}{l} \right)$

with $I = \frac{1}{12} M l^2$

$$\text{So } T = \frac{1}{2} m \dot{x}_3^2 + \frac{1}{8} M \left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1 \dot{x}_2 \right) + \frac{M}{24} \left(\dot{x}_2^2 + \dot{x}_1^2 - 2\dot{x}_1 \dot{x}_2 \right)$$

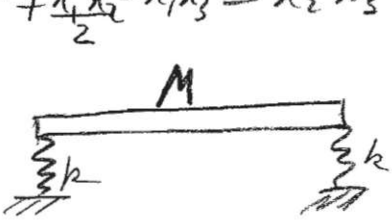
$$= \frac{1}{2} m \dot{x}_3^2 + \frac{1}{6} M \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_1 \dot{x}_2 \right) \quad //$$

PE: $V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k \left(\frac{x_1 + x_2}{2} - x_3 \right)^2$

$$= \frac{1}{2} k \left(x_1^2 + x_2^2 + \frac{x_1^2}{4} + \frac{x_2^2}{4} + \frac{x_1 x_2}{2} - (x_1 + x_2) x_3 + x_3^2 \right)$$

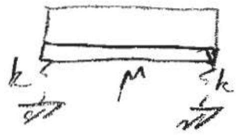
$$= \frac{1}{2} k \left(\frac{5}{4} x_1^2 + \frac{5}{4} x_2^2 + \frac{x_1 x_2}{2} - x_1 x_3 - x_2 x_3 + x_3^2 \right) \quad //$$

(b)(i) $\frac{m}{M} \ll 1$: System is $\approx$



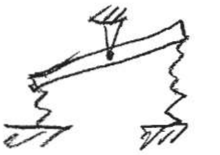
Mode — Frequency

$\begin{Bmatrix} 1 \\ 1 \\ 1+\varepsilon \end{Bmatrix}$



'bounce' of M with m "stuck" on — $\omega_1 \approx \sqrt{\frac{2k}{M}}$ (approx)

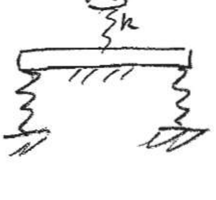
$\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix}$



pure 'pitch'

$\frac{1}{12} M l^2 \ddot{\theta} + \frac{k l^2}{2} \theta = 0 \Rightarrow \omega_2 = \sqrt{\frac{6k}{M}}$

$\begin{Bmatrix} -\varepsilon \\ -\varepsilon \\ 1 \end{Bmatrix}$



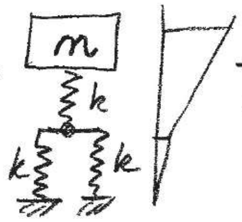
bounce of m

M is ≈ inertial

$\omega_3 = \sqrt{\frac{k}{m}}$ (approx)

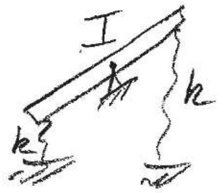
3(b)(ii) $\frac{m}{M} >> 1$

Mode — frequency

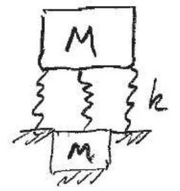
$\begin{Bmatrix} 1/3 \\ 1/3 \\ 1 \end{Bmatrix}$ $k_t \{$ 

$\frac{1}{k_T} = \frac{1}{k} + \frac{1}{2k} \Rightarrow k_t = \frac{2}{3}k \Rightarrow \omega_1 = \sqrt{\frac{2}{3}\frac{k}{m}}$ (approx)

bounce of m on combined stiffness

$\begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$ 

pure pitch $\omega_3 = \sqrt{\frac{6k}{M}}$ (exact, unchanged)

$\begin{Bmatrix} 1 \\ 1 \\ -\varepsilon \end{Bmatrix}$ 

bounce of M (m is inertial) $\omega_2 = \sqrt{\frac{3k}{M}}$ (approx)

(c) for $M/m << 1$ $(x_1\ x_2\ x_3)^T = (1\ 1\ \alpha)^T$

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\cancel{\tfrac{1}{2}}k\left(5/4 + 5/4 + 1/2 - \alpha - \alpha + \alpha^2\right)}{\cancel{\tfrac{1}{2}}\left[m\alpha^2 + \frac{M}{3}(1 + 1 + 1)\right]} = \frac{k(3 - 2\alpha + \alpha^2)}{m\alpha^2 + M}$$

Find exact frequencies by minimizing Rayleigh's quotient

$$\frac{d\omega^2}{d\alpha} = \frac{(m\alpha^2 + M)\cancel{k}(-2 + 2\alpha) - \cancel{k}(3 - 2\alpha + \alpha^2)(2m\alpha)}{(m\alpha^2 + M)^2}$$

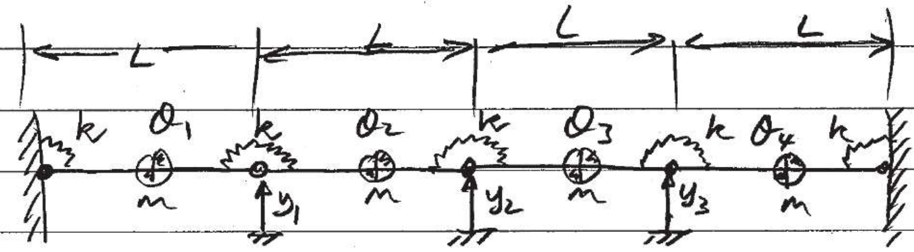
$$\frac{d\omega^2}{d\alpha} = 0 \Rightarrow -2m\alpha^2 - 2M + \cancel{2m\alpha^3} + 2M\alpha - 6m\alpha + 4m\alpha^2 - \cancel{2m\alpha^3} = 0$$

$$\Rightarrow \cancel{2}m\alpha^2 + \alpha(\cancel{2}M - \cancel{6}^{3}m) - \cancel{2}M = 0$$

$$\alpha = \frac{-(M - 3m) \pm \sqrt{(M - 3m)^2 + 4mM}}{2m}$$

$$= \frac{3}{2} - \frac{M}{2m} \pm \sqrt{\left(\frac{M}{2m}\right)^2 - \frac{M}{2m} + 9/4}$$

4(a)



$$V = \tfrac{1}{2}k\left\{\theta_1^2 + (\theta_2-\theta_1)^2 + (\theta_3-\theta_2)^2 + (\theta_4-\theta_3)^2 + \theta_4^2\right\} \quad — (1)$$

$$\theta_1 = y_1/L \qquad \theta_2 = \frac{y_2-y_1}{L} \qquad \theta_3 = \frac{y_3-y_2}{L} \qquad \theta_4 = -y_3/L \quad — (2)$$

$$\therefore V = \frac{1}{2}\frac{k}{L^2}\left\{ y_1^2 + \left[y_2-y_1-y_1\right]^2 + \left[y_3-y_2-y_2+y_1\right]^2 + \left[-y_3-y_3+y_2\right]^2 + y_3^2\right\}$$

$$= \frac{1}{2}\frac{k}{L^2}\left\{6y_1^2 + 6y_2^2 + 6y_3^2 - 8y_1y_2 - 8y_2y_3 + 2y_1y_3\right\}$$

$$= \frac{1}{2}\frac{k}{L^2}\left[y_1\ y_2\ y_3\right]\begin{bmatrix} 6 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 6 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \quad — (3)$$

$$T = \tfrac{1}{2}m\left(\frac{\dot{y}_1}{2}\right)^2 + \tfrac{1}{2}I\dot{\theta}_1^2 + \tfrac{1}{2}m\left(\frac{\dot{y}_2+\dot{y}_1}{2}\right)^2 + \tfrac{1}{2}I\dot{\theta}_2^2$$

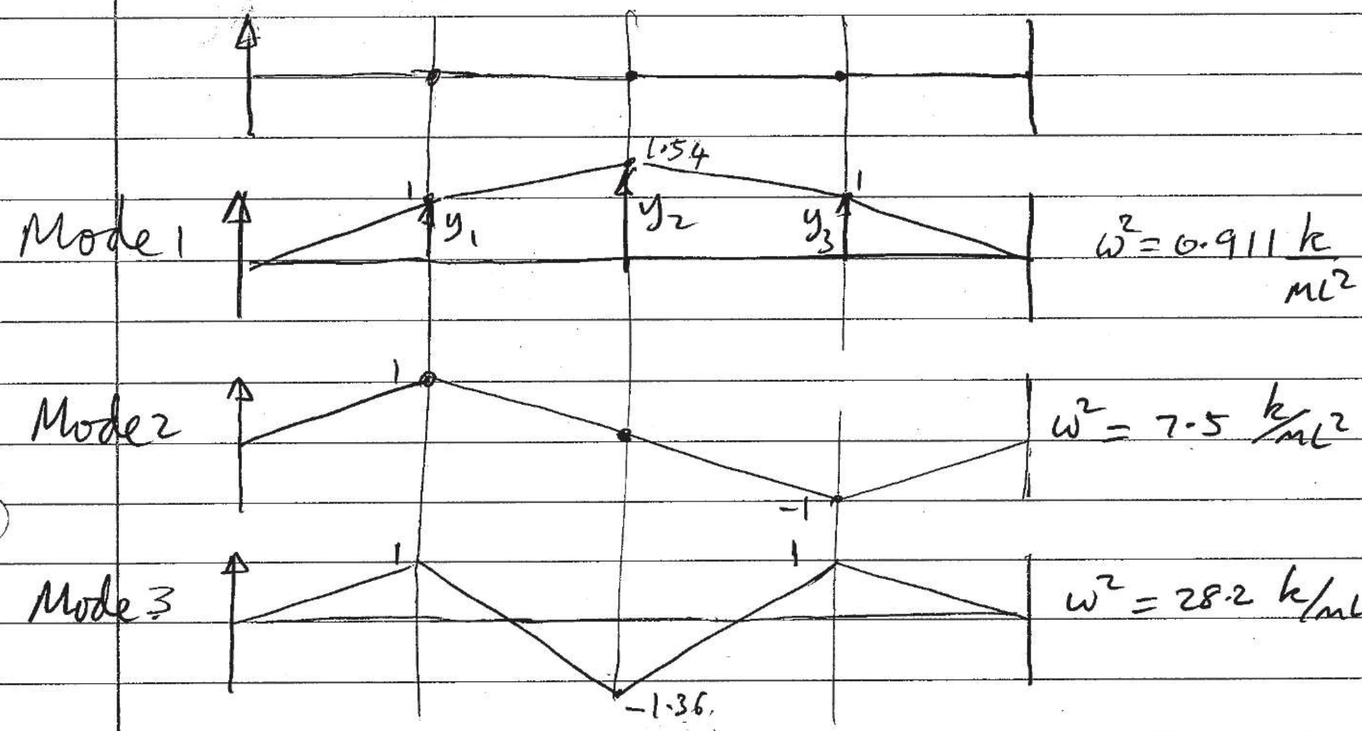
$$+ \tfrac{1}{2}m\left(\frac{\dot{y}_3+\dot{y}_2}{2}\right)^2 + \tfrac{1}{2}I\dot{\theta}_3^2 + \tfrac{1}{2}m\left(\frac{\dot{y}_3}{2}\right)^2 + \tfrac{1}{2}I\dot{\theta}_4^2 \quad — (4)$$

with θ's from (2) & $I = \dfrac{mL^2}{12}$

$$T = \frac{1}{2}m\left\{\frac{2}{3}\dot{y}_1^2 + \frac{2}{3}\dot{y}_2^2 + \frac{2}{3}\dot{y}_3^2 + \frac{1}{3}\dot{y}_1\dot{y}_2 + \frac{1}{3}\dot{y}_2\dot{y}_3\right\}$$

$$= \tfrac{1}{2}m\left[\dot{y}_1\ \dot{y}_2\ \dot{y}_3\right]\begin{bmatrix} 2/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 2/3 \end{bmatrix}\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{Bmatrix} \quad — (5)$$

4 (b) Mode shapes



Let Symmetric modes (1) & (3) be $[1\ \alpha\ 1]^T$

Rayleigh: $\omega^2 = \dfrac{V}{T^*}$

$$= \frac{k}{mL^2}\left[\frac{6y_1^2+6y_2^2+6y_3^2-8y_1y_2-8y_2y_3+2y_1y_3}{\frac{2}{3}y_1^2+\frac{2}{3}y_2^2+\frac{2}{3}y_3^2+\frac{1}{3}y_1y_2+\frac{1}{3}y_2y_3}\right]$$

for $[1\ \alpha\ 1]^T$ $$= \frac{k}{ML^2}\left[\frac{14+6\alpha^2-16\alpha}{4/3+\frac{2}{3}\alpha^2+\frac{2}{3}\alpha}\right] \quad \text{—— (6)}$$

$$\frac{d\omega^2}{d\alpha}=0 \Rightarrow \left(\frac{2}{3}\alpha^2+\frac{2}{3}\alpha+\frac{4}{3}\right)\left(12\alpha-16\right) = \left(6\alpha^2-16\alpha+14\right)\left(\frac{4}{3}\alpha+\frac{2}{3}\right)$$

$\Rightarrow \alpha = \underline{\underline{-1.36,\ 1.54}}$ Exact mode shapes minimise Rayleigh.

$\therefore$ (6) $\Rightarrow \omega^2 = \dfrac{k}{ML^2}\left[\dfrac{14+6(-1.36)^2-16(-1.36)}{4/3+2/3(-1.36)^2+2/3(-1.36)}\right] = 28.2\dfrac{k}{ML^2}$ & $0.911\dfrac{k}{ML^2}$

Hence modes (1) & (3) plotted above.

(c) Mode ②: Write the eigenvalue/vector equation:

$$\left([K]-\omega^2[M]\right)\underset{\sim}{u}=0$$

for 2nd mode, the exact mode shape is $[1\ \ 0\ \ -1]^T$

Using $[K]$ & $[M]$ from ③ & ⑤:

$$\left[\frac{k}{L^2}\begin{bmatrix}6 & -4 & 1\\ - & - & -\\ & - & -\end{bmatrix}-\omega^2 m\begin{bmatrix}2/3 & 1/6 & 0\\ & & \\ & & \end{bmatrix}\right]\begin{Bmatrix}1\\0\\-1\end{Bmatrix}=0$$

$$\Rightarrow \frac{6k}{L^2}-\frac{2}{3}\omega^2 m-\frac{k}{L^2}=0 \quad\Rightarrow \omega^2=\frac{15}{2}\frac{k}{ML^2}$$

$$\therefore\ \omega_2^2=7.5\frac{k}{ML^2}$$ Exact frequency (Exact mode shape)

(d) Input at ①, output at ③

$$\frac{q_k}{F_j}=\sum_{n=1}^{N}\frac{u_k^{(n)}u_j^{(n)}}{\omega_n^2-\omega^2}$$

	Sign
$u_1^{(1)}u_3^{(1)}$	+
$u_1^{(2)}u_3^{(2)}$	−
$u_1^{(3)}u_3^{(3)}$	+