Version JW/3

No changes to paper

EGT2
ENGINEERING TRIPOS PART IIA

Thursday 24 April 2014 9.30 to 11

Module 3C6

VIBRATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C5 Dynamics and 3C6 Vibration data sheet (6 pages).
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) A stretched string with tension P and mass per unit length m can undergo small transverse vibration with displacement $w(x,t)$, where x is distance along the string and t is time. A point mass M is attached to the string at the point $x = a$. Show that the condition

$$M \left.\frac{\partial^2 w}{\partial t^2}\right|_{x=a} = P\left\{ \left.\frac{\partial w}{\partial x}\right|_{x=a+} - \left.\frac{\partial w}{\partial x}\right|_{x=a-} \right\}$$

must be satisfied, and give the second boundary condition that applies at $x = a$. [10%]

(b) A stretched string as in part (a) has length $2L$, and the mass M is attached at its mid-point. The two ends of the string are fixed to rigid supports. Use the governing equation from the Data Sheet together with the various boundary conditions to show that the natural frequencies ω of *symmetric* modes of the weighted string satisfy the equation

$$\tan kL = \frac{2m}{Mk}$$

where $k = \omega\sqrt{m/P}$.

What equation determines the natural frequencies of antisymmetric modes? [40%]

(c) Use a graphical approach to show the behaviour of the solutions of the equations found in part (b). Discuss the limiting cases of very small and very large M. [25%]

(d) For the case when M is very small, explain how Rayleigh's principle can be used to estimate the effect of M on the lowest natural frequency of the string without the added mass. Using the energy expressions given in the Data Sheet, obtain an expression for this Rayleigh estimate. [25%]

2 (a) A beam with a uniform rectangular cross-section is freely pinned to a rigid support at one end, at position $x = 0$, and is free at the other end $x = L$. The width of the beam is b and the thickness h, and it is made of material with Young's modulus E and density ρ. The beam undergoes small-amplitude bending vibrations. Derive an expression whose roots give the natural frequencies. [35%]

(b) Give a graphical construction to estimate the roots of this equation, and hence give approximate expressions for the natural frequencies. Sketch carefully the first three mode shapes. [30%]

(c) Based on the approximate results from part (b), express the second and third non-zero mode frequencies in terms of musical semitones relative to the first non-zero frequency. [15%]

(d) The pinned-free beam from part (a) now hangs freely under gravity. Without calculations, describe carefully what qualitative difference(s) the presence of gravity will make to the governing equation, to the mode shapes and to the natural frequencies.

[20%]

(TURN OVER

3 A simple model of axial vibration of a tapered column consists of four masses connected by four springs as shown in Fig. 1. The masses can move only in the axial direction, with small displacements y_1, y_2, y_3, y_4 as shown.

(a) Write down expressions for the potential and kinetic energy of this system. Ignore gravity. [20%]

(b) Calculate the *static* responses of this system to (i) an axial load applied to the mass m, disregarding the effects of gravity; and (ii) the self-weight of the four masses. [40%]

(c) Use these two static responses with Rayleigh's principle to obtain approximations to the lowest natural frequency of vibration of the system. Explain which of the two estimates is the more accurate and comment on why this might be so. [40%]

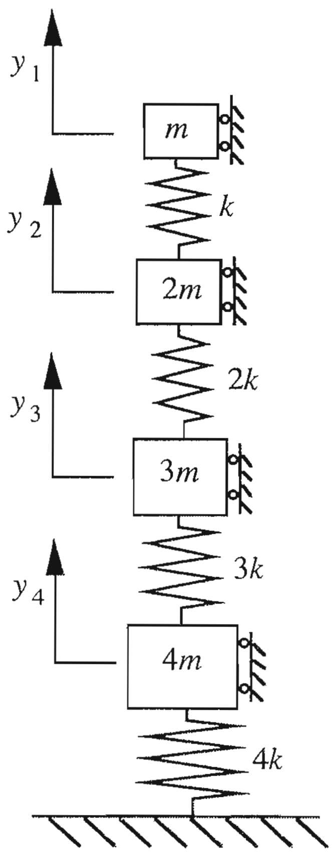

Fig. 1

4 A farmer builds a fence by nailing wooden beams to posts that are embedded in the ground. In plan view, the fence can be idealized as shown in Fig. 2. The fence can be idealized as four uniform, rigid beams of length L and mass m connected with frictionless pin joints at the posts. The posts have lateral stiffness k and the system has no damping. Use the lateral displacements of the posts $y_1, y_2 \dots y_5$ as coordinates.

(a) Write an expression for the kinetic energy of lateral vibration. Hence show that the mass matrix for small vibration of the system can be written in the form:

$$m\begin{bmatrix} A & B & & & \\ B & C & B & & \\ & B & C & B & \\ & & B & C & B \\ & & & B & A \end{bmatrix}$$ [25%]

Determine A, B, and C.

(b) Without calculations, sketch approximately the natural mode shapes for small vibration in order of increasing frequency. Comment on the qualitative features you incorporate in these sketches. [25%]

(c) On a dB scale, sketch the vibration transfer function for displacement of post y_5 due to a sinusoidal force $f = F\sin(\omega t)$ applied to the middle post. Label salient features of your sketch. [20%]

(d) For the particular case of the anti-symmetric modes of the fence, show how the problem can be reduced to one with two degrees of freedom, and hence find an expression for the natural frequencies of the anti-symmetric modes. [30%]

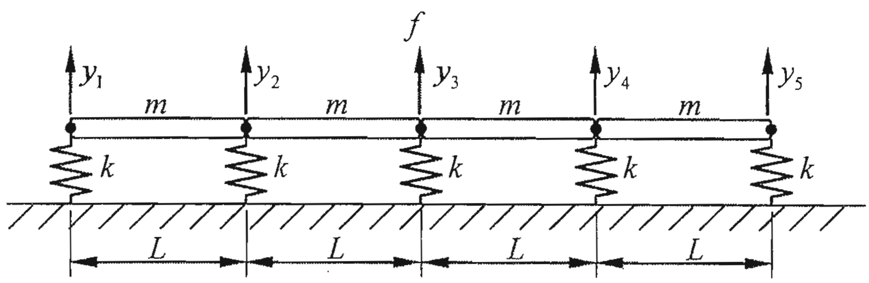

Fig. 2

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Part IIA Data sheet

Module 3C5 Dynamics

Module 3C6 Vibration

DYNAMICS IN THREE DIMENSIONS

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \mathbf{r})\,dm = I\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$I = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $A = \int (y^2 + z^2)\,dm$ $\qquad B = \int (z^2 + x^2)\,dm$ $\qquad C = \int (x^2 + y^2)\,dm$

$D = \int yz\,dm$ $\qquad E = \int zx\,dm$ $\qquad F = \int xy\,dm$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B-C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C-A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A-B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1=\omega_1$ and $\Omega_2=\omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\underline{\ddot{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\underline{\dot{y}}^t M \underline{\dot{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)} \; .$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\,\underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 6 for examples.

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 4 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 4) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\,u_k(x)\,dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\,u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\operatorname{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f} \; .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \ge 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t \; e^{-\omega_n \zeta_n t}$$

for $t \ge 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \ge 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t \; e^{-\omega_n \zeta_n t}$$

for $t \ge 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t<0 \\ 1 & t \ge 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\ge$ the smallest squared frequency;

(2) $\le$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

GOVERNING EQUATIONS FOR CONTINUOUS SYSTEMS

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)(a^4 - b^4)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

Answers

1(b) $\omega = \dfrac{n\pi}{L}\sqrt{\dfrac{P}{m}}, \quad n = 1,2,3\ldots$; (d) $\omega^2 \approx \dfrac{P\pi^2}{4L(mL+M)}$

2(a) $\tan\alpha L = \tanh\alpha L, \quad \omega^2 = \dfrac{Eh^2}{12\rho}\alpha^4$;

(c) 20.35, 33.08 semitones

3(b) Displacement patterns (i) [25, 13, 7, 3]; (ii) [14, 12, 9, 5]

(c) (i) $\omega \approx 0.5117\sqrt{k/m}$; (ii) $\omega \approx 0.4534\sqrt{k/m}$

4 (a) $A = 1/3$; $B = 1/6$; $C = 2/3$

(d) $\omega = \sqrt{\dfrac{6k}{7m}\left(3 \pm \sqrt{2}\right)}$