EGT2
ENGINEERING TRIPOS PART IIA

Friday 30 April 2021 9 to 10.40

Module 3C6

VIBRATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Attachment: 3C5 Dynamics and 3C6 Vibration data sheet (7 pages).
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 A cantilever beam of length L, uniform cross-section of area A and second moment of area I is made of a material with density ρ and Young's modulus E.

(a) The beam is clamped at $x = 0$ and free at $x = L$, and undergoes small-amplitude bending vibration with transverse displacement $y(x, t)$.

(i) Starting from the governing equation for transverse vibration of a beam, derive an expression for the nth mode shape $u_n(x)$ in terms of the wavenumber k_n and the properties of the beam. [20%]

(ii) Find the natural frequencies ω_n of the beam for the first four modes, using a suitable approximation where needed. Express your answer as factors α_n of the first natural frequency ω_1, i.e. in the form $\omega_n = \alpha_n \omega_1$. [10%]

(b) A single degree-of-freedom mass-on-spring is now connected to the beam at $x = L$, with mass m and stiffness k, as shown in Fig. 1. The beam-spring connection is a frictionless pin, and the mass displacement is indicated by $z(t)$. The natural frequency of the uncoupled mass-on-spring is defined as $\Omega = \sqrt{k/m}$.

(i) For the case $k \to 0$, what are the first four natural frequencies of the coupled system? [10%]

(ii) For the case $k \to \infty$, qualitatively describe what you expect for the first four natural frequencies of the coupled system. [10%]

(iii) Derive an expression whose solutions give the natural frequencies of the coupled system. Write your answer in terms of the driving point transfer function $H(L, L, \omega)$ of the uncoupled beam and any other system parameters that are needed. [30%]

(iv) For the case when $\Omega \approx \omega_2$ (i.e. when the natural frequency of the mass-on-spring is tuned to be close to the second natural frequency of the uncoupled beam), use a graphical construction to show the effect of the mass-on-spring on the natural frequencies of the coupled system ω_n'. [20%]

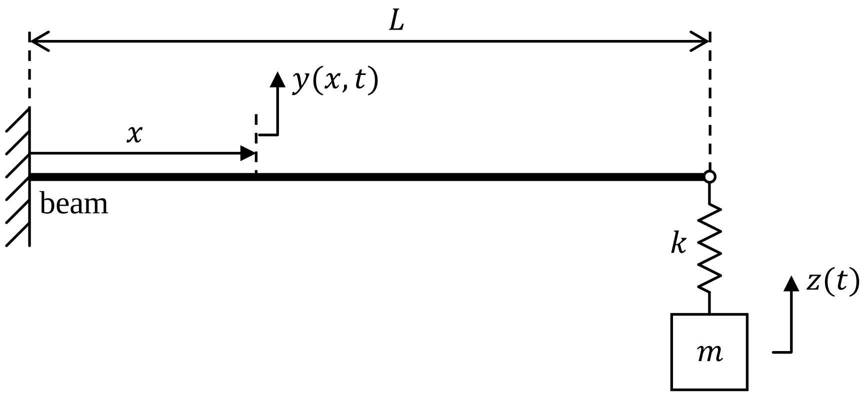

Fig. 1

2 A straight pipe is to be inspected for a possible crack. It is proposed that an axial vibration test be carried out to look for changes in the natural frequencies. The pipe has length L and cross-sectional area A. It is made from a material of density ρ and Young's modulus E. The pipe can undergo small amplitude axial oscillations $y(x,t)$ where x is the distance from one end of the pipe and t is time. The ends of the pipe are free.

(a) For an undamaged pipe:

(i) find expressions for the mode shapes and natural frequencies of the pipe. [10%]

(ii) find a summation expression for the driving point transfer function $G(x,x,\omega)$ from input force to output axial displacement at a distance x from one end of the pipe, allowing for light damping. [20%]

(b) A small crack at a distance $x = a$ from one end of the pipe can be modelled as a local reduction in axial stiffness. The potential energy of the cracked pipe can be written:

$$V = V_0 - \Delta V$$

where V_0 is the potential energy of an undamaged pipe and ΔV is the reduction in potential energy caused by the crack which can be written:

$$\Delta V = \frac{1}{2} C \left(\left. \frac{\partial y}{\partial x} \right|_{x=a} \right)^2$$

C is a constant that corresponds to the reduction in stiffness and the term in brackets indicates the first derivative of displacement evaluated at $x = a$. All other properties of the pipe can be assumed to be unchanged from part (a).

(i) Use Rayleigh's principle to find an approximate expression for the new natural frequency ω_n' of the nth mode of the pipe for a given stiffness reduction C. [30%]

(ii) Sketch the driving point transfer function $G(L,L,\omega)$ for the undamaged pipe of part (a) and when there is a small crack at a distance $x = L/4$ from one end. Highlight which modes are most and least affected by the presence of the crack. [30%]

(iii) Briefly describe how the crack could be detected using a wave-based approach. [10%]

3 Figure 2 shows a uniform rigid beam of mass M and length L, supported by two springs of stiffness k. A second point mass m is attached to the centre of the beam by another spring of stiffness k. The beam can move in the vertical direction and rotate, while the point mass can move in the vertical direction only. The displacements of the ends of the beam and the point mass from equilibrium are denoted x_1, x_2 and x_3 as shown.

(a) Write an expression for the kinetic energy and show that the potential energy is

$$V = \frac{k}{2}\left(\frac{5}{4}x_1^2 + \frac{5}{4}x_2^2 + x_3^2 + \frac{1}{2}x_1x_2 - x_1x_3 - x_2x_3\right).$$

[20%]

(b) Sketch the mode shapes and write down estimates of the natural frequencies for the cases:

(i) $m/M \ll 1$

(ii) $m/M \gg 1$

For each case state which one of these frequencies is exact. [40%]

(c) For the case in part (b)(ii) use Rayleigh's quotient with the mode shape $(x_1, x_2, x_3)^T = (1, 1, \alpha)^T$ to find an exact expression for the remaining two natural frequencies. Compare these frequencies and corresponding mode shapes with your estimates in part (b)(ii). [40%]

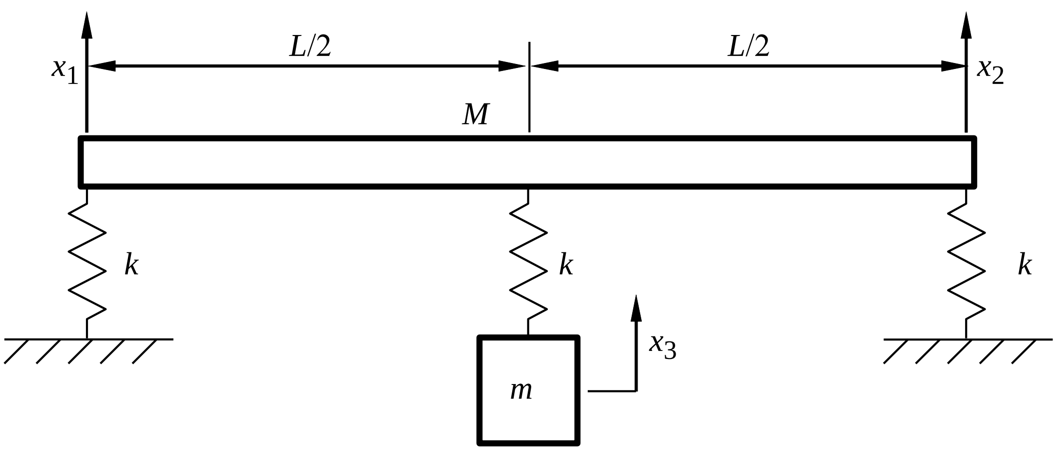

Fig. 2

4 A beam is modelled as four uniform, rigid rods of length L and mass m, joined by frictionless pins at their ends and connected by springs with bending stiffness k, as shown in Fig. 3. The rods are constrained to vibrate in a vertical plane and the displacements of the joints are y_1, y_2 and y_3. Ignore tension in the rods.

(a) Show that for small transverse vibration, the potential energy can be written:

$$V = \frac{k}{L^2}\left\{3y_1^2 + 3y_2^2 + 3y_3^2 - 4y_1y_2 - 4y_2y_3 + y_1y_3\right\}$$

Hence or otherwise determine the stiffness matrix. [10%]

(b) Determine the kinetic energy and the mass matrix. [20%]

(c) Sketch the natural mode shapes. [20%]

(d) Differentiate Rayleigh's quotient to find the natural frequencies and corresponding mode shapes of two symmetric modes. Explain whether or not these frequencies are exact. [20%]

(e) Find another natural frequency from one of your assumed mode shapes using an equation relating eigenvalues and eigenvectors. Explain whether or not this frequency is exact. [10%]

(f) Sketch the amplitude of the displacement y_3 due to a sinusoidal force F_1 on a dB scale. [20%]

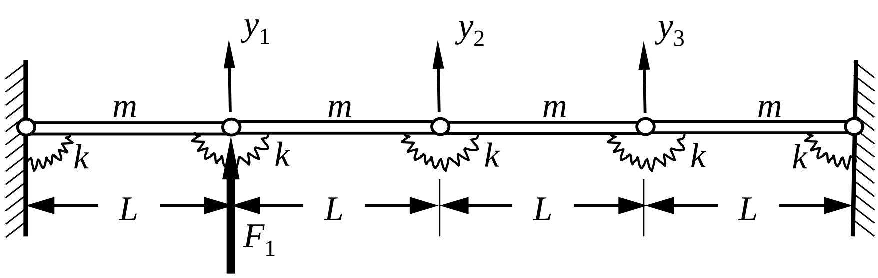

Fig. 3

END OF PAPER

Part IIA Data Sheet

Module 3C5 Dynamics
Module 3C6 Vibration

1 Dynamics in three dimensions

1.1 Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\boldsymbol{v}_\mathrm{G}$, M is the total mass, $\boldsymbol{v}_\mathrm{G}$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\begin{aligned} \boldsymbol{Q}^{(e)} &= (\boldsymbol{r}_\mathrm{G} - \boldsymbol{r}_\mathrm{P}) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_\mathrm{G} \\ &= \dot{\boldsymbol{h}}_\mathrm{P} + \dot{\boldsymbol{r}}_\mathrm{P} \times \boldsymbol{p} \end{aligned}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here $\boldsymbol{h}_\mathrm{P}$ and $\boldsymbol{h}_\mathrm{G}$ are the moments of momentum about P and G respectively, so that for example

$$\begin{aligned} \boldsymbol{h}_P &= \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_\mathrm{P}) \times m_i \dot{\boldsymbol{r}}_i \\ &= \boldsymbol{h}_\mathrm{G} + (\boldsymbol{r}_\mathrm{G} - \boldsymbol{r}_\mathrm{P}) \times \boldsymbol{p} \end{aligned}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) dm = \boldsymbol{I}\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $A = \int (y^2 + z^2) dm \qquad B = \int (z^2 + x^2) dm \qquad C = \int (x^2 + y^2) dm$

$D = \int yz\ dm \qquad E = \int zx\ dm \qquad F = \int xy\ dm$

where all integrals are taken over the volume of the body.

1.2 Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1.1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

1.3 Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= Q_1 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= Q_2 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= Q_3 \end{aligned}$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the"Gyroscope equations"):

$$\begin{aligned} A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 &= Q_1 \\ A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 &= Q_2 \\ C\dot{\omega}_3 &= Q_3 \end{aligned}$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

1.4 Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy and Q_i are the non-conservative generalised forces.

1.5 Hamilton's equations

(a) Basic formulation

The generalized momenta p_i and the Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q})$ are defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \qquad H(\boldsymbol{p}, \boldsymbol{q}) = \sum_i p_i \dot{q}_i - T + V$$

where it should be noted that in the expression for the Hamiltonian the velocities $\dot{q}_i(\boldsymbol{p}, \boldsymbol{q})$ must be expressed as a function of the generalized momenta and the generalized displacements.

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i.$$

(b) Extension topics

The total time derivative of some function $f(\boldsymbol{p}, \boldsymbol{q}, t)$ can be expressed in terms of the Poisson bracket $\{f, H\}$ in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}, \qquad \{f, H\} \equiv \sum_i \left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right].$$

Common forms of Canonical Transform for Hamilton's equations are:

Type	Generating function	1st eqn	2nd eqn	Kamiltonian
1	$G_1(\boldsymbol{q}, \boldsymbol{Q}, t)$	$\boldsymbol{p} = \frac{\partial G_1}{\partial \boldsymbol{q}}$	$\boldsymbol{P} = -\frac{\partial G_1}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_1}{\partial t}$
2	$G_2(\boldsymbol{q}, \boldsymbol{P}, t)$	$\boldsymbol{p} = \frac{\partial G_2}{\partial \boldsymbol{q}}$	$\boldsymbol{Q} = \frac{\partial G_2}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_2}{\partial t}$
3	$G_3(\boldsymbol{p}, \boldsymbol{Q}, t)$	$\boldsymbol{q} = -\frac{\partial G_3}{\partial \boldsymbol{p}}$	$\boldsymbol{P} = -\frac{\partial G_3}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_3}{\partial t}$
4	$G_4(\boldsymbol{p}, \boldsymbol{P}, t)$	$\boldsymbol{q} = -\frac{\partial G_4}{\partial \boldsymbol{p}}$	$\boldsymbol{Q} = \frac{\partial G_4}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_4}{\partial t}$

2 Vibration modes and response

Discrete Systems

1. Equation of motion

The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

2. Kinetic Energy

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

3. Potential Energy

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

4. Natural frequencies and mode shapes

The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

5. Orthogonality and normalisation

$$\mathbf{u}^{(j)T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the ‘quantity’ of the jth mode.

Continuous Systems

1. Equation of motion

The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 3 for examples.

2. Kinetic Energy

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

See Section 3 for examples.

4. Natural frequencies and mode shapes

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 3) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_{j} q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the ‘quantity’ of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n\zeta_n t} \sin\omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1,x_2,t) = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1,x_2,t) \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n} e^{-\omega_n\zeta_n t} \sin\omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t=0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t=0$ at point x_1, the response at point x_2 is

$$h(x_1,x_2,t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1,x_2,t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

2.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\tilde{T}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 3.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.

3 Governing equations for continuous systems

3.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\big(a^4 - b^4\big)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

3.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$\begin{aligned}
\cos\alpha\cosh\alpha + 1 = 0, &\quad \alpha_1 = 1.8751\\
\cos\alpha\cosh\alpha - 1 = 0, &\quad \alpha_1 = 4.7300\\
\tan\alpha - \tanh\alpha = 0, &\quad \alpha_1 = 3.9266
\end{aligned}$$