3C7 2018

Q1 (a) $\nabla^2\psi = -2G\beta$, where β is twist per unit length

Here, $\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 2(B+C)$

$\Rightarrow \beta = -(B+C)/G$ (constant ✓) ✓ (1)

Also require that $\psi = 0$ on boundary

$\Rightarrow \frac{\psi}{A} = 1 + \frac{B}{A}x^2 + \frac{C}{A}y^2 = 0$

on boundary provided $\frac{B}{A} = -\frac{1}{a^2}$ and $\frac{C}{A} = -\frac{1}{b^2}$ (from eqn. of the ellipse)

$\Rightarrow \beta = A\left(\frac{1}{a^2} + \frac{1}{b^2}\right)/G$

Conclusion: ψ satisfies $\nabla^2\psi = -2G\beta$ and $\psi = 0$ on boundary if $B = -A/a^2$ and $C = -A/b^2$, i.e. $\psi = A\left(1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right)$

(b) $T = 2\int_A \psi \, dA$

$= 2A\int_A \left(1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right) dA$

$= 2A\pi ab/2 = \pi ab A$

$\therefore \beta = A\left(\frac{a^2+b^2}{a^2b^2}\right)/G$ (from (1))

$\Rightarrow T = \frac{\pi a^3 b^3}{a^2+b^2} G\beta$

Stress: $\tau_{zy} = -\partial\psi/\partial x = -2Bx = 2Ax/a^2$ (=0 when x=0)

$\tau_{zx} = \partial\psi/\partial y = 2Cy = -2Ay/b^2$

Q1 (c) $T_{hole} = \dfrac{\pi a^3 b^3 (1/64)\, G\beta}{a^2+b^2\,(1/4)} = \dfrac{1}{16} T$

$$T_{hollow} = T - T_{hole} = \frac{15}{16} \frac{\pi a^3 b^3}{a^2+b^2} G\beta$$

Q2 (a) $\phi = C_1 \sin 2\theta + C_2\theta + C_3 \cos 2\theta$

To be an Airy stress function, ϕ must satisfy:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}}_{\text{both zero}} + \underbrace{\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}}\right) = 0$$

$$= -4C_1 \sin 2\theta - 4C_3 \cos 2\theta$$

$$\left.\begin{aligned}
\frac{\partial^2}{\partial r^2}\left(\frac{-4C_1}{r^2}\sin 2\theta\right) &= -24\, C_1 \sin 2\theta / r^4 \\
\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{-4C_1}{r^2}\sin 2\theta\right) &= 8C_1 \sin 2\theta / r^4 \\
\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{-4C_1}{r^2}\sin 2\theta\right) &= 16C_1 \sin 2\theta / r^4
\end{aligned}\right\}\ \text{sum to zero}$$

Similarly for the C_3 term, sum = 0

$\Rightarrow$ Valid Airy stress function.

(b) i) $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ on $\theta = \pm\alpha$

ii) $\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -\frac{4C_1}{r^2}\sin 2\theta - \frac{4C_3}{r^2}\cos 2\theta$

$\sigma_{\theta\theta} = \partial^2\phi/\partial r^2 = 0$

$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{1}{r^2}(2C_1 \cos 2\theta + C_2) - \frac{1}{r^2}(2C_3 \sin 2\theta)$

Consider boundary conditions:

$\theta = \pm\alpha$, $\sigma_{r\theta} = 0 \Rightarrow 2C_1 \cos 2\alpha = -C_2$ and $C_3 = 0$

$\theta = \pm\alpha$, $\sigma_{\theta\theta} = 0$ ✓

Summary: $\sigma_{rr} = \frac{-4C_1}{r^2}\sin 2\theta$

$\sigma_{r\theta} = \frac{2C_1}{r^2}(\cos 2\theta - \cos 2\alpha)$

$\sigma_{\theta\theta} = 0$

Q2 c) $M = \int_{-\alpha}^{\alpha} \sigma_{r\theta} \cdot r \cdot r\,d\theta = \int_{-\alpha}^{\alpha} \sigma_{r\theta}\, r^2\, d\theta$

$$= 2c_1 (\sin 2\alpha - 2\alpha \cos 2\alpha)$$

$$\Rightarrow \quad c_1 = \frac{M}{2(\sin 2\alpha - 2\alpha \cos 2\alpha}$$

$\boxed{\text{Almost all candidates incorrectly attempted to integrate } \sigma_{r\theta}}$

$$\sigma_{rr} = \frac{-2M \sin 2\theta}{r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$$

$$\sigma_{r\theta} = \frac{M(\cos 2\theta - \cos 2\alpha)}{r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$$

$$\sigma_{\theta\theta} = 0$$

d) For α small, simple beam theory would give:

$\sigma = My/I$ where $y = r\theta$

$$I = \frac{(2r\alpha)^3}{12} = \frac{2}{3} r^3 \alpha^3$$

$$\Rightarrow \quad \sigma = \frac{3}{2}\frac{M\theta}{r^2\alpha^3} \quad *$$

$$\sigma_{rr} = \frac{-2M\sin 2\theta}{r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$$

Expand trig functions,

$$= \frac{-2M\left(2\theta - \cancel{\frac{8\theta^3}{3!}}^{\to 0} + O(\theta^5)\right)}{r^2\left(\cancel{2\alpha} \underbrace{- \frac{8\alpha^3}{3!} - \cancel{2\alpha} + 2\alpha \cdot \frac{4\alpha^2}{2!}}_{\text{must include these terms}} + O(\alpha^5)\right)}$$

$$= \frac{12M\theta}{8r^2\alpha^3} \quad * \quad \checkmark$$

Q3 a) 1. Stress state is symmetric about any diameter

2. Shear stress $\sigma_{r\theta} = 0$ along any radial line

3. Stresses σ_{rr} and $\sigma_{\theta\theta}$ are function of r only

4. Isotropic solid

b) From data sheet:

$$\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0 \qquad (2)$$

Eqn (2) automatically satisfied for considered problem, eqn. (1) becomes:

$$\frac{\partial}{\partial r}(r\sigma_{rr}) - \sigma_{\theta\theta} = 0$$

Re-arranging, $r\dfrac{\partial \sigma_{rr}}{\partial r} = \sigma_{\theta\theta} - \sigma_{rr}$

c) Compatibility (data sheet)

$$\frac{\partial}{\partial r}\left(r\frac{\partial \gamma_{r\theta}}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_{\theta\theta}}{\partial r}\right) - r\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$$

For thin circular disk, $\gamma_{r\theta} = 0$, and ε_{rr} and $\varepsilon_{\theta\theta}$ do not depend on θ, therefore:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_{\theta\theta}}{\partial r}\right) - r\frac{\partial \varepsilon_{rr}}{\partial r} = 0$$

$$\Rightarrow r^{\cancel{2}}\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} + 2\cancel{r}\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \cancel{r}\frac{\partial \varepsilon_{rr}}{\partial r} = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial \varepsilon_{\theta\theta}}{\partial r}\right) - \frac{\partial \varepsilon_{\theta\theta}}{\partial r} + 2\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{\partial \varepsilon_{rr}}{\partial r} = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial \varepsilon_{\theta\theta}}{\partial r}\right) = \frac{\partial}{\partial r}(\varepsilon_{rr} - \varepsilon_{\theta\theta})$$

$$\Rightarrow r\frac{\partial \varepsilon_{\theta\theta}}{\partial r} = \varepsilon_{rr} - \varepsilon_{\theta\theta}$$

Q3 d) $\sigma_{rr} = A - \frac{B}{r^2} - \frac{E\alpha}{r^2}\int_0^r rT\,dr$

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} + \frac{E\alpha}{r^2}\int_0^r rT\,dr - E\alpha T$$

$$(T = T(r) - T_0)$$

e) $\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(u_r r)}{\partial r}\right] = (1+\nu)\alpha\frac{\partial(T-T_0)}{\partial r}$

Integrate once,

$$\frac{1}{r}\frac{\partial(ur)}{\partial r} = (1+\nu)\alpha(T-T_0) + 2C_2$$

$$\Rightarrow \frac{\partial(ur)}{\partial r} = (1+\nu)\alpha(T-T_0)r + 2C_2 r$$

Integrate again,

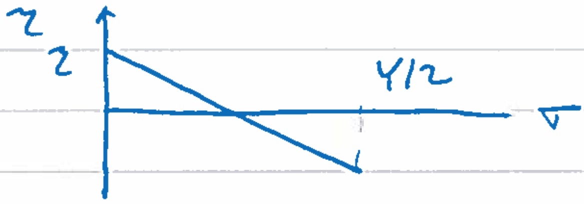
$$ur = (1+\nu)\alpha\int_0^r r(T-T_0)\,dr + C_2 r^2 + C_1$$

$$\Rightarrow u = C_2 r + \frac{C_1}{r} + (1+\nu)\frac{\alpha}{r}\int_0^r r(T-T_0)\,dr$$

Q4 a) Stress tensor (matrix)

$$\underline{\underline{\sigma}} = \begin{bmatrix} Y/2 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Mohr circle:



$$|\sigma_2 - \sigma_1| = 2\sqrt{\tau^2 + Y^2/16} = Y$$

$$\tau^2 + Y^2/16 = Y^2/4$$

$$\Rightarrow \tau = \pm \frac{\sqrt{3}}{4} Y$$

Principal strain increments (from data sheet)

$$d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$$

b) i) $\sigma_{rr} = A - B/r^2$, $\sigma_{\theta\theta} = A + B/r^2$

Plane strain: $0 = \varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta})$

$$\Rightarrow \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

Far field boundary condition:

As $r \to \infty$ $\sigma_{rr} = \sigma_{\theta\theta} = 0 \Rightarrow A = 0$

At $r = a$: $\sigma_{rr} = -p \Rightarrow B = pa^2$

$$\Rightarrow \sigma_{rr} = -pa^2/r^2 , \quad \sigma_{\theta\theta} = pa^2/r^2$$

Since $\sigma_{rr} < \sigma_{zz} < \sigma_{\theta\theta}$, yield occurs when $\sigma_{\theta\theta} - \sigma_{rr} = Y$

Q4 b) (cont)

i)

$$2pa^2/r^2 = Y \implies p = Yr^2/(2a^2) \qquad (1)$$

$$\implies \sigma_{rr} = \frac{-Ya^2}{2r^2}, \quad \sigma_{\theta\theta} = \frac{Ya^2}{2r^2}$$

$$(\sigma_{rr} = -Y/2, \ \sigma_{\theta\theta} = Y/2 \ @ \ r=a)$$

c) Equilibrium must always be satisfied,

$$\therefore \frac{\partial \sigma_{rr}}{\partial r} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} = \frac{Y}{r}$$

Consider a plastic tube with radius c nested in an elastic hole. The interface pressure between them is $-p_i$.

$$\int_{-p}^{-p_i} \frac{\partial \sigma_{rr}}{\partial r} dr = Y \int_a^c \frac{dr}{r}$$

$$p - p_i = Y \ln\left(\frac{c}{a}\right)$$

$$p_i = p - Y \ln\left(\frac{c}{a}\right)$$

The is still elastic for $r > c$, hence from (1)

$$p_i = Y/2$$

Equating (1) and (2),

$$p = Y\left(\frac{1}{2} + \ln\left(\frac{c}{a}\right)\right)$$

GNW