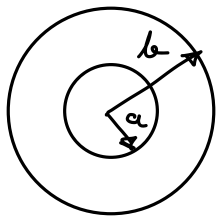
Gib 3C7 (21/22)

Q1

(a)




$a = 20mm$

$b = 40mm$ $\qquad \mu = 0.09$

$l = 60mm$

(i) $Q = \mu p 2\pi a l a$

$$p = \frac{Q}{2\pi \mu a^2 l} = 36.8 \text{ MPa}$$

(ii) In plane stress

$$\sigma_{rr} = A - \frac{B}{r^2}, \quad \sigma_{\theta\theta} = A + \frac{B}{r^2}, \quad \sigma_{zz} = 0$$

$$\Rightarrow \varepsilon_{\theta\theta} = \frac{u}{r} = \frac{1}{E}\left[A + \frac{B}{r^2} - \nu\left(A - \frac{B}{r^2}\right)\right]$$

$$u = \frac{r}{E}\left[A(1-\nu) + \frac{B}{r^2}(1+\nu)\right]$$

Rotor: @ $r = b$, $\sigma_{rr} = 0 \Rightarrow A = \frac{B}{b^2}$

@ $r = a$ $\sigma_{rr} = -p \Rightarrow -p = A - \frac{B}{a^2}$

$$\Rightarrow A = \frac{pa^2}{b^2 - a^2}; \quad B = \frac{pa^2b^2}{b^2 - a^2}$$

@ $r = a$ $\quad u_R = \frac{pa}{E}\left[\frac{a^2(1-\nu) + b^2(1+\nu)}{b^2 - a^2}\right]$

<u>Shaft</u> $\sigma_{rr} = \sigma_{\theta\theta} = -p$ as $B = 0$. & $A = -p$

@ $r = a$ $\quad u_s = -\dfrac{pa(1-\nu)}{E}$

$$\Rightarrow \delta = u_R - u_s = \frac{pa(1-\nu)}{E}\left[1 + \frac{a^2 + b^2(1+\nu)/(1-\nu)}{b^2 - a^2}\right]$$

$$= \frac{2pa}{E}\,\frac{b^2}{b^2-a^2}$$

$$\delta = 9.3\,\mu m$$

(iii) Critical point is at $r = a$

$\sigma_{rr} = -p, \quad \sigma_{\theta\theta} = A + \dfrac{B}{a^2} = p\left(\dfrac{b^2+a^2}{b^2-a^2}\right), \quad \sigma_{zz} = 0$

$\Rightarrow$ max. principal stress $= \sigma_{\theta\theta} = 61.3$ MPa.

(b) Design is suitable if $\tau_{max} < \dfrac{Y}{3}$

From a(iii) $\quad |\sigma_{\theta\theta} - \sigma_{rr}| > |\sigma_{\theta\theta} - \sigma_{zz}| > |\sigma_{rr} - \sigma_{zz}|$

$$\tau_{max} = \left.\frac{\sigma_{\theta\theta} - \sigma_{rr}}{2}\right|_{r=a} = 49\text{ MPa}$$

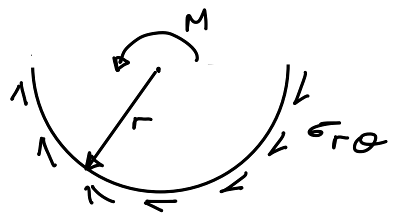
$\tau_{max} = 49$ MPa $< \dfrac{Y}{3} = 80$ MPa $\Rightarrow$ design suitable

Q2

(a) $$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -\frac{1}{r^2}\left(2A\sin 2\theta\right)$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = -\frac{A}{r^2}\left(1+\cos 2\theta\right)$$




$$M = \int_{-\frac{\pi}{2}}^{\pi/2} \sigma_{r\theta} r^2 \, d\theta \implies A = \frac{M}{\pi}$$

ie $$\sigma_{rr} = -\frac{2M}{\pi r^2}\sin 2\theta \; ; \; \sigma_{\theta\theta} = 0$$

$$\sigma_{r\theta} = -\frac{M}{\pi r^2}\left(1+\cos 2\theta\right)$$

(b)

$$x = r\cos\theta \; ; \; y = r\sin\theta$$

$$\Rightarrow r^2 = x^2 + y^2 \quad \therefore \left.\frac{\partial r}{\partial x}\right|_y = \cos\theta \; \& \; \left.\frac{\partial r}{\partial y}\right|_x = \sin\theta$$

$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \left.\frac{\partial \theta}{\partial x}\right|_y = -\frac{y}{r^2} = -\frac{\sin\theta}{r} \quad \& \quad \left.\frac{\partial \theta}{\partial y}\right|_x = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

$$\therefore \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial \psi}{\partial r}\sin\theta + \frac{\partial \psi}{\partial \theta}\frac{\cos\theta}{r}$$

(c)

$$\phi_1 = \phi(x,y) - \phi(x, y+a) \quad - \text{superposition}$$

For $a \to 0$

$$\phi_1 = \phi - \left[\phi + a\frac{\partial \phi}{\partial y}\right]$$

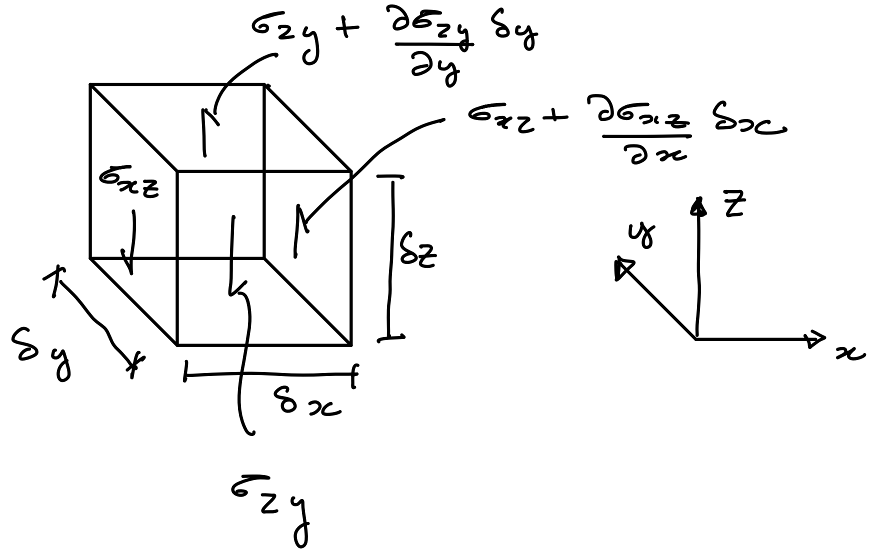
$$= -a\frac{\partial \phi}{\partial y}$$

But $$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r}\sin\theta + \frac{\partial \phi}{\partial \theta}\frac{\cos\theta}{r}$$

$$\Rightarrow \phi_1 = -\frac{2Ma}{\pi}\cos^3\theta$$

Q3

(a)




Force equilibrium in $z$-direction dictates

$$\frac{\partial \sigma_{xz}}{\partial x}\delta x\,\delta y\,\delta z + \frac{\partial \sigma_{zy}}{\partial y}\delta y\,\delta x\,\delta z = 0$$

$$\Rightarrow \quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0$$

(b)

$$\sigma_{zx} = G\left[-\beta y + \frac{\partial w}{\partial x}\right]$$

$$\frac{\partial \sigma_{zx}}{\partial x} = G\frac{\partial^2 w}{\partial x^2}$$

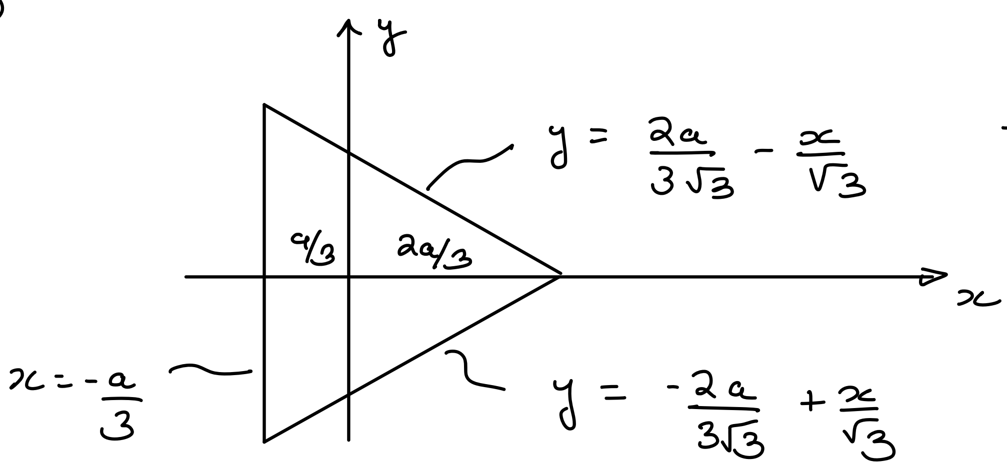
$$\sigma_{zy} = G\left[\beta x + \frac{\partial w}{\partial y}\right]$$

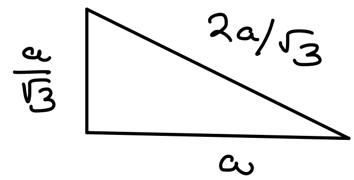
$$\frac{\partial \sigma_{zy}}{\partial y} = G\frac{\partial^2 w}{\partial y^2}$$

$\Rightarrow$ Equilibrium dictates

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow \nabla^2 w = 0$$

(c)




$\Rightarrow \quad \psi = 0$ on perimeter ✓

We also require $\nabla^2 \psi = -2G\beta$

$$\frac{\partial \psi}{\partial x} = c'\left(2x - \frac{3x^2}{a} + \frac{3y^2}{a}\right); \quad \frac{\partial^2 \psi}{\partial x^2} = 2c' - \frac{6c'x}{a}$$

$$\frac{\partial \psi}{\partial y} = c'\left(2y + \frac{6xy}{a}\right); \quad \frac{\partial^2 \psi}{\partial y^2} = 2c' + \frac{6c'x}{a}$$

$$\Rightarrow \nabla^2 \psi = 4c' \Rightarrow 4c' = -2G\beta$$

$$c' = -\frac{G\beta}{2}$$

where $c' = \frac{ca}{3}$ ie $c = \frac{-3G\beta}{2a}$

(d)

$$w = B\beta(y^3 - 3x^2 y)$$

$$\frac{\partial^2 w}{\partial y^2} = 6B\beta y \; ; \; \frac{\partial^2 w}{\partial x^2} = -6B\beta y \implies \nabla^2 w = 0 \checkmark$$

$$\sigma_{xz} = -G\beta\left[y + 6Bxy\right] \; ; \; \sigma_{yz} = G\beta\left[x + 3By^2 - 3Bx^2\right]$$

But $\sigma_{xz} = \dfrac{\partial \psi}{\partial y} = -G\beta\left[\dfrac{3xy}{a} + y\right]$

$$\sigma_{yz} = -\frac{\partial \psi}{\partial x} = G\beta\left[x - \frac{3x^2}{2a} + \frac{3y^2}{2a}\right]$$

Comparing co-efficients $B = \dfrac{1}{2a}$

Q4

(a) The upper bound calculation is performed in the following steps

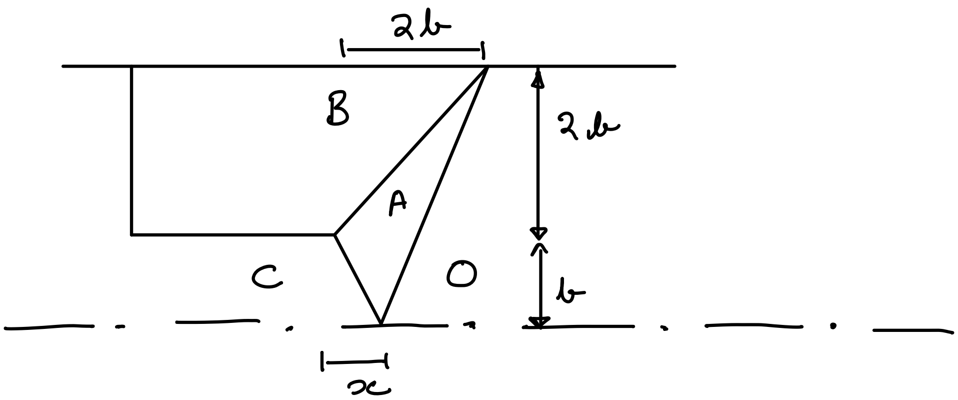
(i) propose a kinematically admissible deformation mode that satisfies the displacement boundary conditions

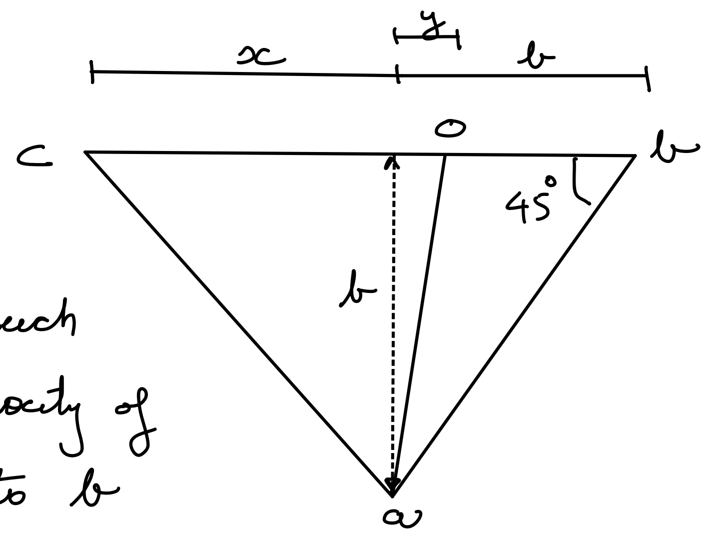
(ii) Equate the internal plastic work associated with the deformation mode to the external work to calculate the collapse load.

If the proposed mode is the exact solution the stresses associated with this mode satisfy eqilibrium.

(b)

(i)






Choose the scale of the velocity diagram such that the vertical velocity of A is proportional to b

$$ob = (b - y)\,; \qquad oc = x + y$$

Volume conservation of metal $\Rightarrow 2ob = oc$

$$\Rightarrow 2b - 2y = x + y$$

$$y = \frac{2b}{3} - \frac{x}{3}$$

$$\Rightarrow F v_B = k\left[ l_{AC} + l_{OA}\, v_{OA} \right]$$

$$F\left[ b - \frac{1}{3}(2b - x) \right] = k\left[ \sqrt{x^2 + b^2}\sqrt{x^2 + b^2} + \sqrt{(2b - x)^2 + 9b^2}\,\frac{\sqrt{(2b - x)^2 + 9b^2}}{3} \right]$$

$$\Rightarrow \frac{F}{k} = 4\left[ \frac{x^2 - bx + 4b^2}{x + b} \right] \quad \& \quad P = 2F$$

(ii)

$$\frac{1}{4k}\frac{dF}{dx} = \left(\frac{2x-b}{x+b}\right) - \frac{x^2-bx+4b^2}{(x+b)^2}$$

$$= \frac{2x^2+xb-b^2-x^2+bx-4b^2}{(x+b)^2}$$

$$= \frac{x^2+2bx-5b^2}{(x+b)^2} = 0$$

$$\Rightarrow \quad x = \frac{-2b \pm \sqrt{4b^2+20b^2}}{2}$$

$$= -b + \sqrt{6}\,b$$

$$x = b(\sqrt{6}-1)$$

$$\& \quad F_{min} = 4kb\left(\frac{12-3\sqrt{6}}{\sqrt{6}}\right) = 7.6\,kb$$

Total $2F_{min} = \underline{15.2\,kb} = P_{min}$

(iii) The calculation to include friction along the sloping surfaces will use the energy balance equation

$$\frac{P}{2} v_B = k \left[ l_{AC} + l_{OA} \, v_{OA} \right] + f \, l_{AB} \, \gamma_{ab}$$

where $f$ is the friction stress along the sloping surfaces.