EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 26 April 2022   9.30 to 11.10

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
3C7 formulae sheet (2 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A turbine rotor has the form of a cylinder of outer radius 40 mm and length 60 mm with a central circular hole of radius 20 mm. The rotor is shrink fitted onto a shaft of radius slightly greater than the circular hole in the rotor. The rotor and the shaft are made from the same alloy steel, with Young's modulus $E = 210$ GPa, Poisson's ratio $\nu = 0.3$, and uniaxial tensile yield strength $Y = 240$ MPa.

(a) The assembly is to transmit a torque $Q = 500$ Nm and no slip is to occur between the shaft and the rotor. Under plane stress conditions and assuming no yielding, determine:

(i) the minimum interfacial pressure between the rotor and the shaft if the friction coefficient between the rotor and the shaft is $\mu = 0.09$; [20%]

(ii) the corresponding minimum interference fit δ between the shaft and the rotor; [50%]

(iii) the maximum principal stress in the assembly (you may neglect the contribution to the stress field due to the torque). [10%]

(b) If the maximum shear stress in the assembly is limited to $Y/3$, comment on whether the above design of the rotor assembly is suitable. [20%]

2 (a) In polar co-ordinates (r, θ), a Airy stress function ϕ is given by

$$\phi = A\left(\theta + \frac{\sin 2\theta}{2}\right),$$

where A is a constant. This stress function has been proposed to determine the stress field in an elastic half-space subjected to a surface couple M as shown in Fig. 1a. Calculate the stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ at any (r, θ) and hence determine A in terms of M. [40%]

(b) For any function $\psi(x, y)$ in Cartesian co-ordinates (x, y) show that

$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial r}\sin\theta + \frac{\partial\psi}{\partial\theta}\frac{\cos\theta}{r},$$

using the usual relations $x = r\cos\theta$ and $y = r\sin\theta$. [20%]

(c) We now wish to calculate the stresses for the case of two equal and opposite surface couples acting a small distance a apart as shown in Fig. 1b. Using superposition along with the stress function from (a) show that, for a vanishingly small a, the appropriate stress function is given by

$$\phi_1 = -\frac{2Ma}{\pi r}\cos^3\theta.$$

[Hint: Use the co-ordinate transformation law obtained in (b).] [40%]

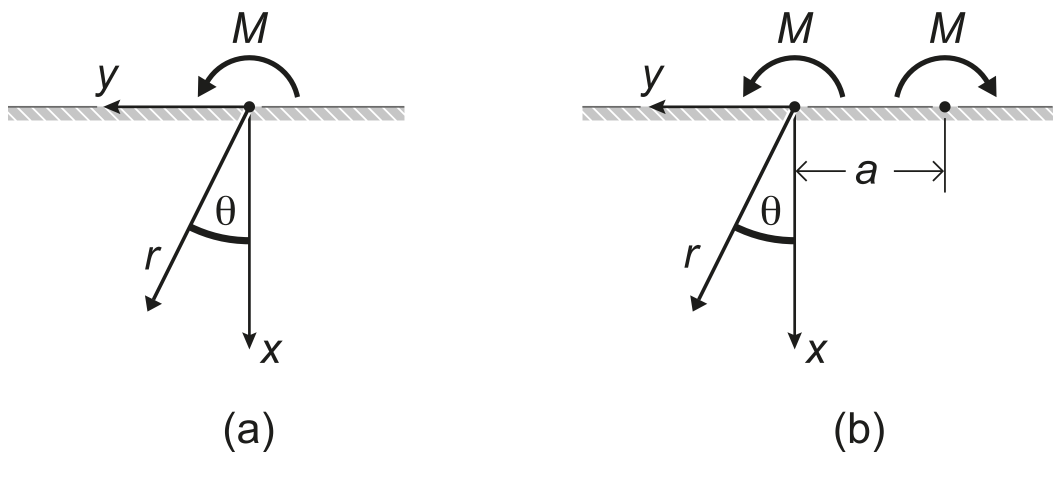

Fig. 1

3 (a) Derive the equilibrium equation in the $z-$direction in terms of the shear stress components σ_{zy} and σ_{zx} with all other stress components equal to zero. [20%]

(b) For the case of a torsion of a shaft made from an isotropic elastic material, the strain components are given in terms of the warping function $w(x, y)$ and the twist β per unit length of the shaft by

$$\gamma_{zx} = -\beta y + \frac{\partial w}{\partial x}, \quad \text{and} \quad \gamma_{zy} = \beta x + \frac{\partial w}{\partial y}.$$

Hence show that the warping function satisfies $\nabla^2 w = 0$. [10%]

(c) The function

$$\begin{aligned} \Psi &= C\left(x + \frac{a}{3}\right)\left(y + \frac{x}{\sqrt{3}} - \frac{2a}{3\sqrt{3}}\right)\left(y - \frac{x}{\sqrt{3}} + \frac{2a}{3\sqrt{3}}\right) \\ &= \frac{Ca}{3}\left[x^2 + y^2 - \frac{1}{a}\left(x^3 - 3xy^2\right) - \frac{4a^2}{27}\right], \end{aligned}$$

is being considered as a Prandtl stress function to investigate the elastic torsion of a prismatic bar. The cross-section of the bar is an equilateral triangle of side $2a/\sqrt{3}$ with the origin of the co-ordinate system coincident with the centroid of the cross-section, as shown in Fig. 2. Show that this function represents a valid choice and hence determine the constant C in terms of the twist β per unit length of the shaft and the shear modulus G. [30%]

(d) The function

$$w = B\beta\left(y^3 - 3x^2 y\right),$$

can be used as a warping function for this shaft with the triangular cross-section. Calculate the constant B in terms of the dimension a. [40%]

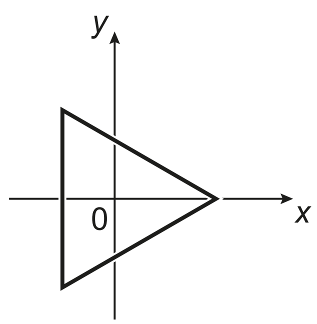

Fig. 2

4 (a) Briefly describe the calculation of the collapse load of a structure using the upper bound theorem of plasticity. Under what circumstances will be the stress associated with an upper bound solution satisfy equilibrium? [20%]

(b) An apparatus for the plane strain "back extrusion" of a perfectly plastic metal with shear yield strength k is shown in Fig. 3. The thickness of the metal slab is reduced from $6b$ to $2b$ in the apparatus under the action of the pair of rigid dies with 45^{o} sloping surfaces.

(i) Using the tangential velocity discontinuity lines shown as dashed lines in Fig. 3 calculate the extrusion force P per unit depth into the page in terms of the variable x. You may assume negligible friction between the metal and all contacting surfaces. [50%]

(ii) Determine the optimum extrusion force. [20%]

(iii) Without performing any further calculations briefly describe how the above calculation can be altered to include the effect of friction along the sloping surfaces. [10%]

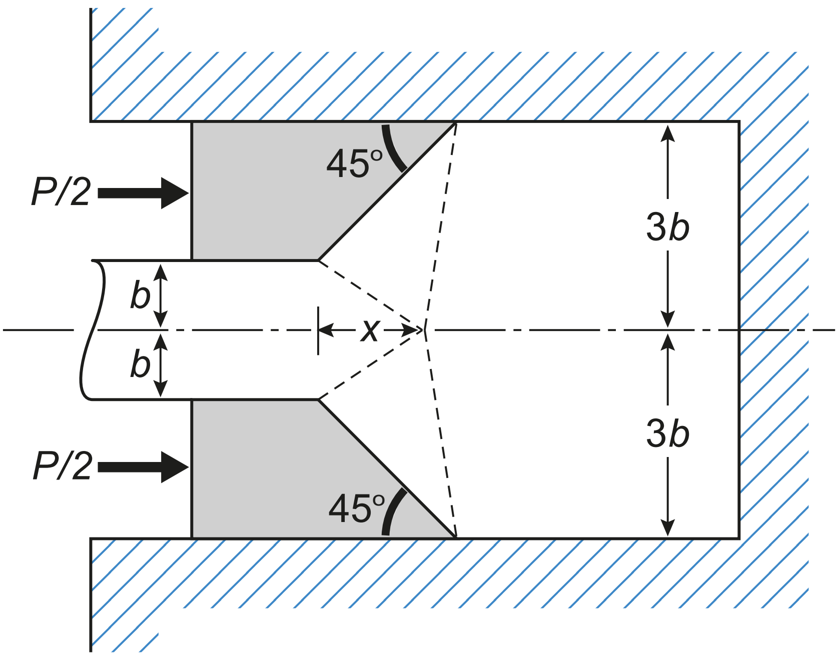

Fig. 3

END OF PAPER

THIS PAGE IS BLANK

Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \bar{E} = \dfrac{E}{1-\nu^2}\ ;\ \bar{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \bar{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial \varepsilon_{rr}}{\partial r} + \dfrac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$
or (in elasticity with no thermal strains or body forces)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\} \times \left\{\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{\partial \psi}{\partial y}, \quad \sigma_{zy}\ (= \tau_y) = -\dfrac{\partial \psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi \, dA$

Governing equation for elastic torsion: $\nabla^2 \psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^{T} [D] \underset{\sim}{\varepsilon} \, dV$, $W = \underset{\sim}{P}^{T} \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1 \sigma_P^2 + I_2 \sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}} \left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}} \left\{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\right\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$