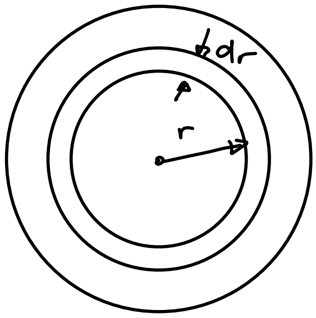
(3C7 crib 2023)

Q1.

(a)



Torque in elemental circumference

$$\delta T = G\beta r(2\pi r\,dr)r$$

$$T = \int_0^{D/2} \delta T = 2\pi G\beta \int_0^{D/2} r^3 dr$$

$$= 2\pi G\beta \left[\frac{r^4}{4}\right]_0^{D/2}$$

$$= \frac{\pi G\beta D^4}{32}$$

(b) The slope of ψ equals shear stress. For axisymmetric problem $\psi = f(r)$ =>

$$\frac{\partial\psi}{\partial r} = -G\beta r$$

$$\psi = -\frac{G\beta r^2}{2} + c$$

Choose $\psi = 0$ on $r = \frac{D}{2}$ =>

$$0 = -\frac{G\beta D^2}{8} + c$$

$$\psi = G\beta\left[\left(\frac{D}{2}\right)^2 - r^2\right]$$

(c)(i) $\nabla^2\psi = \left[\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\right]\psi$

$$\frac{\partial\psi}{\partial r} = -G\beta r \; ; \; \frac{\partial^2\psi}{\partial r^2} = -G\beta \, , \; \frac{\partial^2\psi}{\partial\theta^2} = 0$$

$$\nabla^2\psi = -G\beta - G\beta = -2G\beta$$

(ii) $\int \psi \, dA = \int_0^{D/2} \psi \, 2\pi r \, dr$

$$= \int_0^{D/2} \left[\frac{\pi G\beta}{4} D^2 r - \pi G\beta r^3\right] dr$$

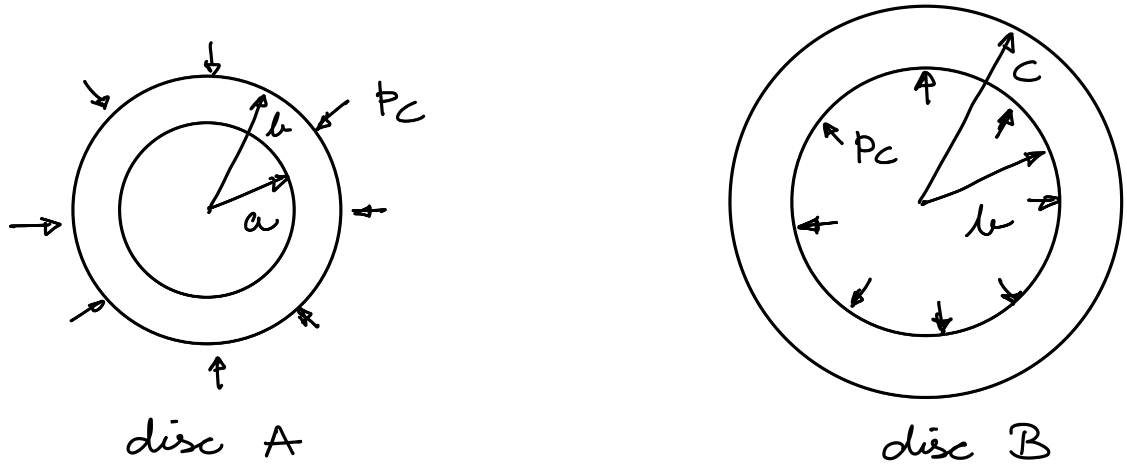
$$= \left[\frac{\pi G\beta D^2}{8} r^2 - \frac{\pi G\beta}{4} r^4\right]_0^{D/2}$$

$$= \frac{\pi G\beta D^4}{64} = \frac{T}{2}$$ ✓

(d) (i) Ψ will not be significantly affected by the slot $\Rightarrow$ stiffness changes negligibly

(ii) However, at corners $\nabla \Psi \to \infty$ $\Rightarrow$ large local stress concentration.

Q2.



Disc A $\quad \sigma_{rr} = A - \frac{B}{r^2}$ & $\sigma_{\theta\theta} = A + \frac{B}{r^2}$

$\sigma_{rr} = 0$ @ $r = a$ & $= -p_c$ @ $r = b$

$$\sigma_{rr} = \frac{-p_c b^2}{b^2 - a^2}\left(1 - \frac{a^2}{b^2}\right); \quad \sigma_{\theta\theta} = \frac{-p_c b^2}{b^2 - a^2}\left(1 + \frac{a^2}{b^2}\right)$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right) = -\frac{p_c b^2}{E(b^2 - a^2)}\left[1 + \frac{a^2}{r^2} - \nu + \frac{\nu a^2}{r^2}\right]$$

Disc B $\quad \sigma_{rr} = -p_c$ @ $r = b$ & $= 0$ @ $r = c$

$$\sigma_{rr} = \frac{p_c b^2}{c^2 - b^2}\left(1 - \frac{c^2}{r^2}\right); \quad \sigma_{\theta\theta} = \frac{p_c b^2}{c^2 - b^2}\left(1 + \frac{c^2}{r^2}\right)$$

$$\varepsilon_{\theta\theta} = \frac{p_c b^2}{E(c^2-b^2)} \left[1 + \frac{c^2}{r^2} - \nu + \frac{\nu c^2}{r^2} \right]$$

(b) at $r = b$ for disc A

$$u^A = u(r=b) = (r\,\varepsilon_{\theta\theta})_{r=b}$$

$$= \frac{p_c b}{E} \left(\nu - \frac{b^2/a^2+1}{b^2/a^2-1} \right)$$

at $r = b$ for disc B

$$u^B = \frac{p_c b}{E} \left(\frac{c^2/b^2+1}{c^2/b^2-1} + \nu \right)$$

$$\Rightarrow \delta = u^B - u^A$$

$$\frac{p_c}{E} = \frac{\delta}{b} \left[\frac{b^2/a^2+1}{b^2/a^2-1} + \frac{c^2/b^2+1}{c^2/b^2-1} \right]^{-1}$$

(c) disc A $\quad \left| \sigma_{rr} - \sigma_{\theta\theta} \right|_{r=a} = \dfrac{2 p_c}{1 - a^2/b^2}$

disc B $\quad \left| \sigma_{rr} - \sigma_{\theta\theta} \right|_{r=b} = \dfrac{2 p_c}{1 - b^2/c^2}$

Thus for simultaneous yielding $\quad \dfrac{b}{c} = \dfrac{a}{b}$

(d) For internal pressure only

$\sigma_{rr} = -p$ @ $r = a$; $\sigma_{rr} = 0$ @ $r = c$

$$\Rightarrow \quad \sigma_{rr} = \frac{p a^2}{c^2 - a^2}\left(1 - \frac{c^2}{r^2}\right) ; \quad \sigma_{\theta\theta} = \frac{p a^2}{c^2 - a^2}\left(1 + \frac{c^2}{r^2}\right)$$

Complete field

$$\left.\begin{aligned} \sigma_{rr} &= \frac{-p_c b^2}{b^2 - a^2}\left(1 - \frac{a^2}{b^2}\right) + \frac{p a^2}{c^2 - a^2}\left(1 - \frac{c^2}{r^2}\right) \\ \sigma_{\theta\theta} &= \frac{-p_c b^2}{b^2 - a^2}\left(1 + \frac{a^2}{b^2}\right) + \frac{p a^2}{c^2 - a^2}\left(1 + \frac{c^2}{r^2}\right) \end{aligned}\right\} \quad a \le r \le b$$

$$\left.\begin{aligned}
\sigma_{rr} &= \frac{p_c b^2}{c^2 - b^2}\left(1 - \frac{c^2}{r^2}\right) + \frac{p a^2}{c^2 - a^2}\left(1 - \frac{c^2}{r^2}\right) \\
\sigma_{\theta\theta} &= \frac{p_c b^2}{c^2 - b^2}\left(1 + \frac{c^2}{r^2}\right) + \frac{p a^2}{c^2 - a^2}\left(1 + \frac{c^2}{r^2}\right)
\end{aligned}\right\} \quad b \leq r \leq c$$

Q3.

(a) $\phi = Ar^2 + Br^2 \ln r + C \ln r + D\theta$

$$\frac{\partial \phi}{\partial r} = 2Ar + 2Br\ln r + Br + \frac{C}{r}$$

$$\frac{\partial^2 \phi}{\partial r^2} = 2A + 2B\ln r + 3B - \frac{C}{r^2}$$

$$\frac{\partial \phi}{\partial \theta} = D \quad ; \quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{D}{r^2}$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\Rightarrow \sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$= 2A + B(2\ln r + 1) + \frac{C}{r^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 2A + B(2\ln r + 3) - \frac{C}{r^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = \frac{D}{r^2}$$

(b) $\sigma_{r\theta} = 0$ on all boundaries $\Rightarrow D = 0$

$\sigma_{rr} = 0$ @ $r = a$

$$\Rightarrow \quad 2A + B + 2B \ln a + \frac{C}{a^2} = 0$$

$$\Rightarrow \quad (2A + B)a^2 + 2Ba^2 \ln a + C = 0 \qquad - (1)$$

$\sigma_{rr} = 0$ @ $r = b$

$$\Rightarrow \quad (2A + B)b^2 + 2Bb^2 \ln b + C = 0 \qquad - (2)$$

(2) − (1)

$$\Rightarrow \quad 2B(b^2 \ln b - a^2 \ln a) = (2A + B)(a^2 - b^2)$$

$$2A(a^2 - b^2) = 2B\left[\frac{b^2 - a^2}{2} + b^2 \ln b - a^2 \ln a\right]$$

$$A = B\left[b^2 \ln b - a^2 \ln a - \frac{1}{2}\right] \qquad - (3)$$

(c) $$-M = \int_a^b \sigma_{\theta\theta}\, r\, dr$$

$$-M = \int_a^b \left(2Ar + 3Br + 2Br\ln r - \frac{C}{r}\right) dr$$

Consider $$I = \int_a^b r \ln r \, dr$$

$$= \left[\frac{r^2}{2}\ln r\right]_a^b - \int_a^b \frac{r^2}{2}\frac{1}{r}\, dr$$

$$= \frac{b^2}{2}\ln b - \frac{a^2}{2}\ln a - \left[\frac{r^2}{4}\right]_a^b$$

$$I = \frac{1}{2}\left(b^2 \ln b - a^2 \ln a\right) - \frac{1}{4}\left(b^2 - a^2\right)$$

$$\Rightarrow -M = \frac{2A+3B}{2}\left(b^2 - a^2\right) - C\ln\frac{b}{a} + B\left(b^2 \ln b - a^2 \ln a\right)$$

$$- \frac{B}{2}\left(b^2 - a^2\right)$$

$$M = -(A+B)(b^2 - a^2) + C \ln\frac{b}{a} - B(b^2 \ln b - a^2 \ln a) \quad — (4)$$

(d) We have $D = 0$

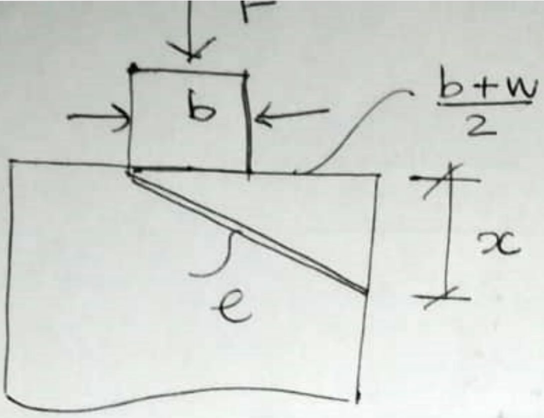
(3) gives $B = f(A)$

Substitute $B = f(A)$ in (1) to get $C = g(A)$

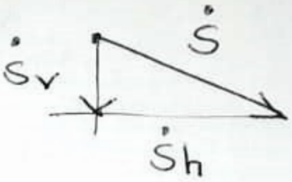
Then (4) can be written as

$M =$ function (A, a, b) which implies we have $A =$ function (M, a, b) & consequently we also have B & C as functions of (M, a, b).

Q4.



$$e = \frac{\sqrt{4x^2 + (b+w)^2}}{2}$$



$$\frac{\dot{S}_v}{\dot{S}} = \frac{x}{e} = \frac{2x}{\sqrt{4x^2+(b+w)^2}} \Rightarrow \dot{S}_v = \dot{S}\, \frac{2x}{\sqrt{4x^2+(b+w)^2}}$$

$$W_{ext} = F \cdot \dot{S}_v = F \cdot \dot{S} \cdot \frac{2x}{\sqrt{4x^2+(b+w)^2}}$$

$$W_{int} = k \cdot \dot{S} \cdot \ell = k \cdot \dot{S} \cdot \frac{\sqrt{4x^2+(b+w)^2}}{2}$$

$$F \cancel{\dot{S}} \frac{2x}{\sqrt{4x^2+(b+w)^2}} = k \cancel{\dot{S}} \cdot \frac{\sqrt{4x^2+(b+w)^2}}{2}$$

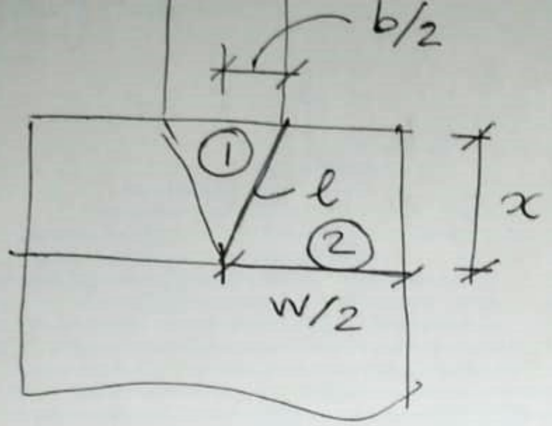
$$F = \frac{k}{4}\, \frac{4x^2+(b+w)^2}{x}$$

$$\frac{\partial F}{\partial x} = \frac{k}{4} \cdot \frac{8x^2 - 4x^2 - (b+w)^2}{x^2} = \frac{k}{4} \cdot \frac{4x^2 - (b+w)^2}{x^2} = 0$$

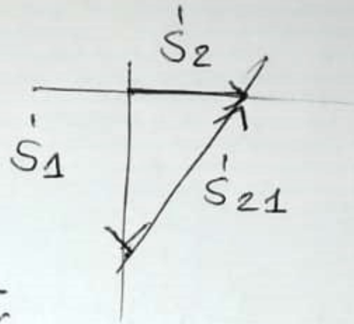
$$x = \frac{b+w}{2}$$

$$F = \frac{k}{4} \cdot \frac{(b+w)^2 + (b+w)^2}{(b+w)} \cdot 2 = \frac{k}{} (b+w) = kb\left(1 + \frac{w}{b}\right)$$

$$\boxed{F = kb\left(1 + \frac{w}{b}\right)}$$ mode 1



$$l = \frac{\sqrt{4x^2+b^2}}{2}$$



$$\frac{\dot{S}_2}{\dot{S}_1} = \frac{b}{2x} \Rightarrow \dot{S}_2 = \dot{S}_1 \frac{b}{2x}$$

$$\frac{\dot{S}_{21}}{\dot{S}_1} = \frac{l}{x} \Rightarrow \dot{S}_{21} = \dot{S}_1 \cdot \frac{l}{x} = \dot{S}_1 \cdot \frac{\sqrt{4x^2+b^2}}{2x}$$

$$W_{ext} = F \cdot \dot{S}_1$$

$$W_{int} = 2\left[k \cdot \frac{w}{2} \cdot \dot{S}_1 \frac{b}{2x} + k \cdot \frac{\sqrt{4x^2+b^2}}{2} \cdot \dot{S}_1 \frac{\sqrt{4x^2+b^2}}{2x}\right]$$

$$F \cancel{\dot{S}_1} = 2k \cancel{\dot{S}_1} \left[\frac{wb}{4x} + \frac{4x^2+b^2}{4x}\right] = \frac{k}{2} \frac{wb+4x^2+b^2}{x}$$

$$\frac{\partial F}{\partial x} = \frac{k}{2} \frac{8x^2 - wb - 4x^2 - b^2}{x^2} = \frac{k}{2} \frac{4x^2 - wb - b^2}{x^2} = 0$$

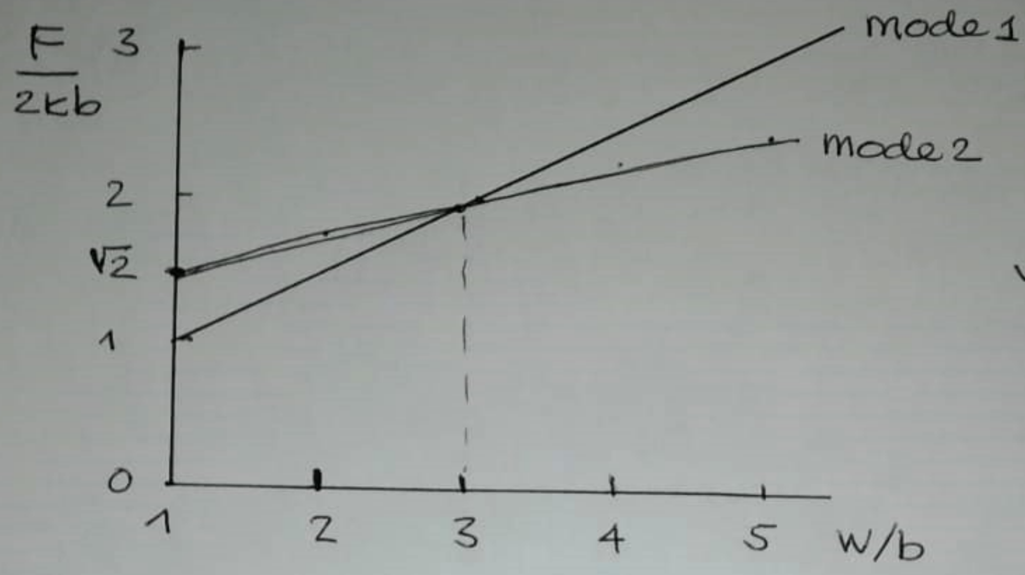
$$x^2 = \frac{b^2+wb}{4} \qquad x = \frac{\sqrt{b^2+wb}}{2}$$

$$F = \frac{k}{\cancel{2}} \cdot \frac{wb + (b^2+wb) + b^2}{\sqrt{b^2+wb}} \cdot \cancel{2} = k \cdot 2 \cdot \frac{(b^2+wb)}{\sqrt{b^2+wb}} =$$

$$= 2k\sqrt{b^2+wb} = 2kb\sqrt{1+\frac{w}{b}}$$

$$\boxed{F = 2kb\sqrt{1+\frac{w}{b}}}$$ mode 2

③



$w/b > 1$

$$\frac{F}{2kb} = \frac{F}{\sigma_Y b} = \frac{1}{2}\left(1 + w/b\right) \quad \text{mode 1}$$

$$\frac{F}{2kb} = \frac{F}{\sigma_Y b} = \sqrt{1 + w/b} \quad \text{mode 2}$$

"critical" $w/b = 3$

either from graph, or from:

$$\frac{1}{2}\left(1 + \frac{w}{b}\right) = \sqrt{1 + w/b} \qquad w/b = y$$

$$1 + y = 2\sqrt{1+y}$$

$$1 + y^2 + 2y = 4 + 4y$$

$$y^2 - 2y - 3 = 0 \qquad y_{1,2} = \frac{2 \mp \sqrt{4+12}}{2} \begin{cases} -1 & \times \\ 3 & \checkmark \end{cases}$$