Version VSD/3

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 25 April 2023   9.30 to 11.10

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
3C7 formulae sheet (2 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A shaft with a uniform circular cross-section of diameter D is made from an isotropic elastic material with shear modulus G. The shaft is subjected to a torque T which causes a twist β per unit length.

(a) The shear stress at a radial location r in the shaft is $\tau = G\beta r$. Hence calculate the torque T in terms of G, β and D. [20%]

(b) Determine an appropriate Prandtl stress function ψ for this problem. [20%]

(c) Show that ψ from part (b):

(i) satisfies the governing equation $\nabla^2\psi = -2G\beta$ [20%]

(ii) satisfies the equilibrium equation $T = 2\int_A \psi \, dA$, where A is the cross-sectional area of the shaft. [20%]

(d) A small rectangular slot is cut in the shaft as shown in cross-section view Fig. 1. Discuss the consequences of this slot on (i) the stiffness T/β and (ii) maximum stress in the shaft. [20%]

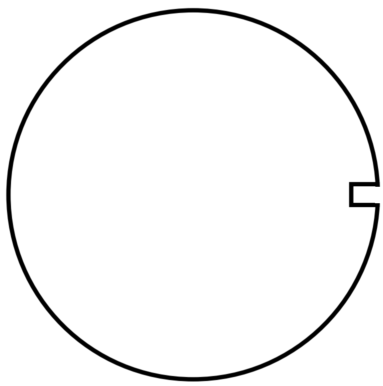


Fig. 1

2 A compound cylindrical disc comprises two concentric discs nested inside each other. Prior to assembly, the outer radius b of the inner disc (with inner radius a) is larger by an amount δ than the inner radius of the outer disc (with outer radius c). After assembly, a contact pressure p_c is developed between the two discs. The discs are made from the same material with Young's modulus E and Poisson's ratio ν. You may assume that plane stress conditions prevail for the compound disc.

(a) Derive expressions for the circumferential strain of the compound disc. [25%]

(b) Derive an expression for the contact pressure p_c. [35%]

(c) Derive the geometric condition for both discs to yield simultaneously at the interface. You may assume that the discs yield following the Tresca yield criterion. [20%]

(d) Assuming that the discs remain elastic, determine the complete stress field within the compound disc in terms of p_c. [20%]

3 Figure 2 shows a curved beam of constant cross-section. The outer and inner surfaces of the beam are arcs of radii b and a, respectively with the arcs subtending an angle $\pi/2$. The ends of the beam are subjected to a bending moment M as indicated. An Airy stress function in the polar co-ordinate system (r, θ)

$$\phi = Ar^2 + Br^2 \ln r + C \ln r + D\theta$$

where A, B, C and D are constants is proposed to analyse the stress state in this beam.

(a) Calculate the stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ in terms of the constants A, B, C and D. [20%]

(b) State the boundary conditions on σ_{rr} and $\sigma_{r\theta}$ and hence determine A and D in terms of B. [30%]

(c) Using the applied moment boundary condition, write an expression for the moment in terms constants A, B, C and D. [40%]

(d) Without further calculations explain how all the constants A, B, C and D may be written in terms of the geometric parameters (a, b) and the loading M. [10%]

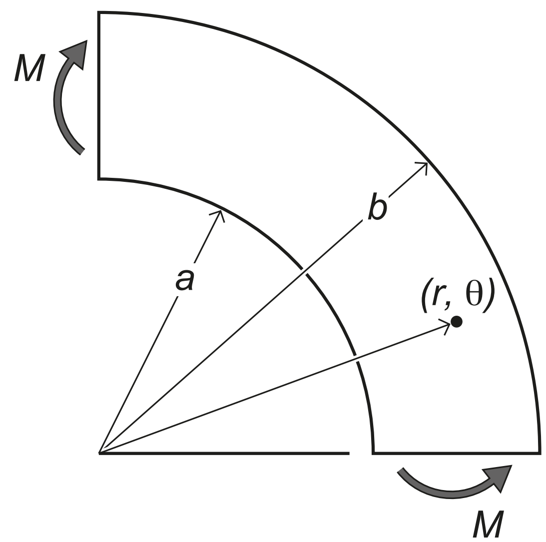


Fig. 2

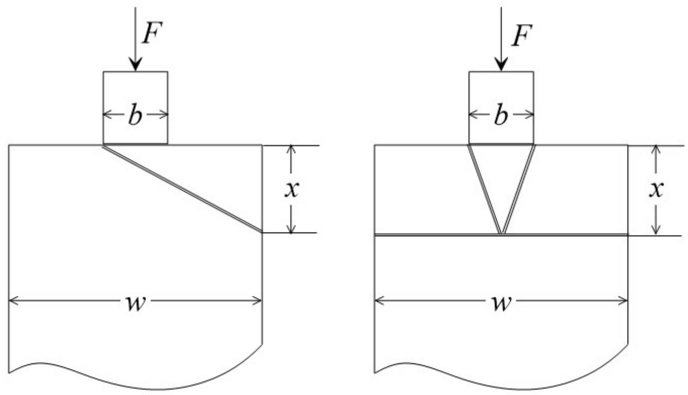
4 (a) Explain briefly why the Upper Bound theorem is useful in modelling metal indentation and forming processes. [20%]

(b) A long, smooth, rigid indenter is pressed by a force F per unit length into a block of rigid-perfectly plastic material with a tensile yield strength σ_Y. Assume the Tresca yield criterion. The width of the indenter and of the block are b and w, respectively, with $w/b > 1$. Figure 3 shows two simple mechanisms in terms of a geometric parameter x as indicated. Assuming plane strain conditions:

(i) Work out an upper bound solution for the indentation force F assuming that the collapse mode is that shown in Fig. 3(a). [30%]

(ii) Work out an upper bound solution for the indentation force F assuming that the collapse mode is that shown in Fig. 3(b). [30%]

(iii) Plot the normalised indentation force, $F/(b\sigma_Y)$, as a function of the width ratio, w/b, for the two collapse modes in Fig. 3 and identify the width ratio at which the collapse mode in Fig. 3(b) becomes the preferred collapse mode. [20%]



(a) (b)

Fig. 3

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Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\displaystyle\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\displaystyle\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \bar{E} = \dfrac{E}{1-\nu^2}\ ;\ \bar{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \bar{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity with no thermal strains or body forces)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{\partial \psi}{\partial y}, \quad \sigma_{zy}\ (= \tau_y) = -\dfrac{\partial \psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi \, dA$

Governing equation for elastic torsion: $\nabla^2 \psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^{T} [D] \, \underset{\sim}{\varepsilon} \, dV$, $W = \underset{\sim}{P}^{T} \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1 \sigma_P^2 + I_2 \sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}} \left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}} \left\{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2 \right\}^{1/2}$

7. Yield criteria and flow rules

<u>Tresca</u>

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

<u>von Mises</u>

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$