EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 23 April 2024   9.30 to 11.10

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
3C7 formulae sheet (2 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The boundaries $x = 0$ and $y = 0$ of a thin, linear elastic quarter-space as shown in Fig. 1, are loaded by pressure distributions $p(y)$ on boundary $x = 0$ and $q(x)$ on boundary $y = 0$. These pressures are polynomial functions of the form

$$\begin{aligned} p(y) &= p_0 + p_1 y + p_2 y^2 + \dots \\ q(x) &= q_0 + q_1 x + q_2 x^2 + \dots \end{aligned}$$

where $p_0, p_1, \dots$ and $q_0, q_1, \dots$ are constants.

(a) Show that with the shear stress $\sigma_{xy} = 0$ everywhere in the quarter-space and $q(x) = 0$, the loading is given by $p(y) = p_0 + p_1 y$. [30%]

(b) Derive the most general form of $p(y)$ and $q(x)$ that results in the absence of shear stress in any direction throughout the quarter-space. [20%]

(c) The quarter-space is now loaded by the normal pressures $q(x) = 0$ and $p(y) = p_0$ in addition to a spatially uniform shear stress $\sigma_{xy} = p_0/2$ on the surfaces $x = 0$ and $y = 0$.

(i) Determine the stresses σ_{xx}, σ_{yy} and σ_{xy} at any location (x, y) within the quarter-space. [25%]

(ii) Hence calculate the principal stresses and corresponding principal directions. [25%]

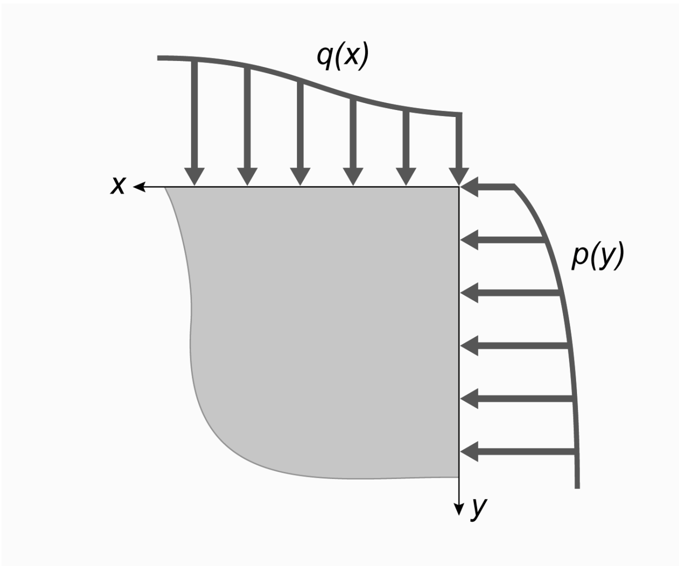

Fig. 1

2 Consider a scalar function $\phi(r,\theta) = Ar^2\theta^m$ in polar coordinates where A and m are constants.

(a) Find all values of m such that ϕ is a valid Airy stress function. [20%]

(b) Consider a thin sheet in the form of a half-space over the region $0 \le \theta \le \pi$ as shown in Fig. 2. We analyse this half-space using the Airy stress function $\phi = Ar^2\theta$.

(i) Determine the tractions along the edges $\theta = 0$ and $\theta = \pi$. [20%]

(ii) Derive expressions for the stresses σ_{xx}, σ_{yy} and σ_{xy} at a location (x, y) in the half-space. [25%]

(iii) Using the above solutions and superposition, derive an expression for the shear stress σ_{xy} at a location (x, y) due to a normal pressure p applied on the surface $y = 0$ over the patch $-a \le x \le a$. [35%]

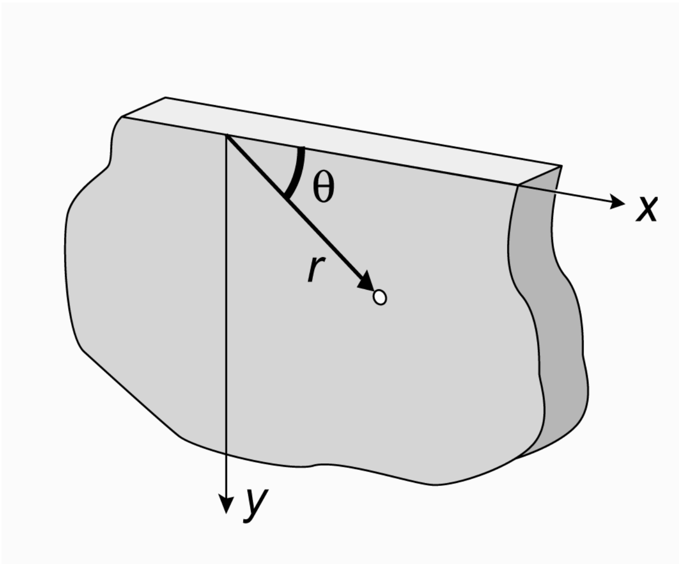

Fig. 2

3 A long shaft (along the $z-$ direction), with an almost square cross-section in the $x-y$ plane, is shown in Fig. 3. The boundaries of the cross-section are defined in terms of a parameter c by

$$x^4 - 6x^2y^2 + y^4 + 5c^2(x^2 + y^2) - 6c^4 = 0.$$

The shaft is made from a linear elastic material with shear modulus G, and is subjected to a torque Q about the $z-$ axis.

(a) Show that $\phi = \beta\left[x^4 - 6x^2y^2 + y^4 + 5c^2(x^2 + y^2) - 6c^4\right]$ is a suitable Prandtl stress function and hence express β in terms of G, c and the twist per unit length α of the shaft. [25%]

(b) Determine the shear stresses σ_{xz} and σ_{yz} on the diagonal $x = y$ and use symmetry arguments to comment on the relation between σ_{xz} and σ_{yz}. How do these values of σ_{xz} and σ_{yz} at $(x, y) = (c, c)$ differ from those for a perfectly square cross-section? [35%]

(c) Calculate σ_{yz} along $y = 0$ and hence determine its maximum value. [15%]

(d) Determine the torsional stiffness Q/α by approximating the shape to be a square of side $2c$ but using the stress function given above. [25%]

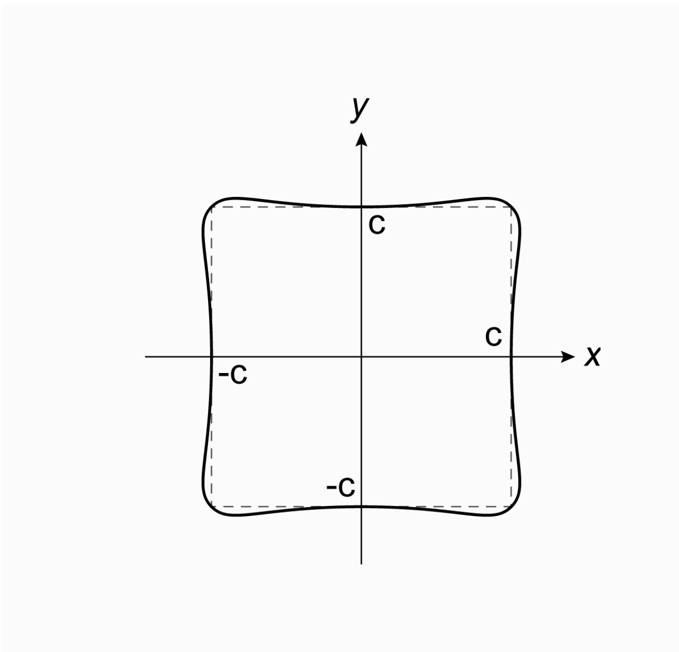

Fig. 3

4 An infinitely large thin sheet containing a circular hole with a radius a is subjected to a remote radial tension $\sigma_{rr} = \lambda$ and a tension $\sigma_{rr} = \lambda/5$ at the hole boundary, where λ is a positive number; see Fig. 4. The sheet is made from an isotropic elastic perfectly-plastic material that obeys the Tresca yield criterion and has a uniaxial tensile yield strength Y.

(a) Determine the stresses σ_{rr} and $\sigma_{\theta\theta}$ in terms of λ assuming that the material remains within its elastic limit. [30%]

(b) Determine λ when first yield occurs. [20%]

(c) When λ is increased, an annular plastic zone of radius c extends concentrically from the edge of the hole. Determine the relationship between λ and the plastic zone radius c. [50%]

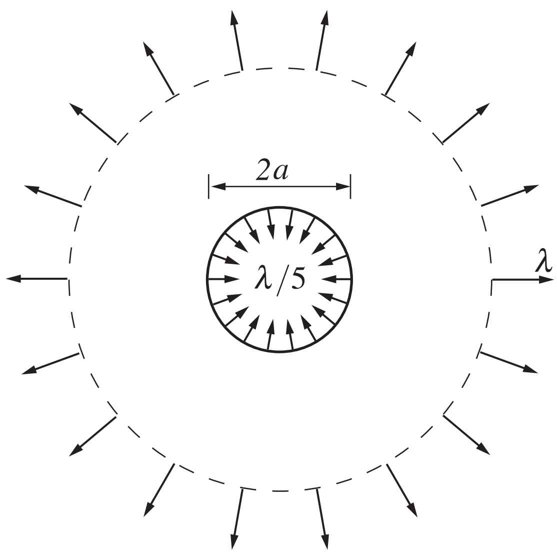

Fig. 4

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Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \bar{E} = \dfrac{E}{1-\nu^2}\ ;\ \bar{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \bar{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial \varepsilon_{rr}}{\partial r} + \dfrac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$
or (in elasticity with no thermal strains or body forces)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\} \times \left\{\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{\partial \psi}{\partial y}, \quad \sigma_{zy}\ (= \tau_y) = -\dfrac{\partial \psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi \, dA$

Governing equation for elastic torsion: $\nabla^2 \psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D] \underset{\sim}{\varepsilon} \, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1 \sigma_P^2 + I_2 \sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}} \left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}} \left\{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\right\}^{1/2}$

7. Yield criteria and flow rules

<u>Tresca</u>

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

<u>von Mises</u>

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}} .$$

3C7 numerical answers

Question 1

(a) $p(y) = p_0 + p_1 y$

(b)

(c) (i) $\left(\sigma_{xx}, \sigma_{yy}, \tau_{xy}\right) = (-p_0, 0, p_0)$

(ii) $\sigma_1 = -p_0/2(\sqrt{2}+1)$; $\sigma_2 = -p_0/2(\sqrt{2}-1)$

Question 2:

(a) $m = 0{,}1$

Question 3:

Question 4:

(b) $\lambda = 5Y/9$