3C7 crib for 2014

1)

a)

$$\sigma_{\theta\theta} = \frac{d}{dr}(r\sigma_{rr}) + \rho\omega^2 r^2$$

$$r\sigma_{rr} = \frac{\rho\omega^2}{32}(3D^2 r - 12r^3) + \frac{\nu\rho\omega^2}{32}(D^2 r - 4r^3)$$

$$\frac{d}{dr}(r\sigma_{rr}) = \frac{\rho\omega^2}{32}(3D^2 - 36r^2) + \frac{\rho\nu\omega^2}{32}(D^2 - 12r^2)$$

$$= \sigma_{\theta\theta} - \rho\omega^2 r^2 \quad \checkmark \quad \text{(in equilibrium)}$$

Check for compatibility

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta})$$

$$= \frac{\rho\omega^2}{32E}(3D^2 - 12r^2) + \frac{\nu\rho\omega^2}{32E}(\overbrace{D^2 - 4r^2 - 3D^2 + 4r^2}^{-2D^2})$$

$$- \frac{\nu^2\rho\omega^2}{32E}(D^2 - 12r^2)$$

$$u = r\varepsilon_{\theta\theta} = \frac{r}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr})$$

$$= \frac{\rho\omega^2}{32E}(3D^2 r - 4r^3) + \frac{\nu\rho\omega^2}{32E}(D^2 r - 12r^3 - 3D^2 r + 12r^3)$$

$$- \frac{\nu^2\rho\omega^2}{32E}(D^2 r - 4r^3)$$

$$\frac{du}{dr} = \frac{\rho\omega^2}{32E}\left(3D^2 - 12r^2\right) + \frac{\nu\rho\omega^2}{32E}\left(-2D^2\right) - \frac{\nu^2\rho\omega^2}{32}\left(D^2 - 12r\right.$$

$$= \varepsilon_{rr} \quad \text{ie compatibility } \checkmark$$

Boundary conditions

$$\sigma_{rr}\Big|_{r=0} = \frac{3D^2\rho\omega^2}{32} + \frac{\nu D^2\rho\omega^2}{32} = \sigma_{\theta\theta}\Big|_{r=0}$$

$$\& \quad \sigma_{rr}\Big|_{r=\frac{D}{2}} = \frac{\rho\omega^2}{32}\left(3D^2 - 3D^2\right) + \frac{\nu\rho\omega^2}{32}\left(D^2 - D^2\right) = 0$$

(b) σ_{rr} & $\sigma_{\theta\theta} > 0$ for all r. This is plane stress

$\Rightarrow$ active yield condition

$$\underset{\sigma_{\theta\theta}}{\overset{\sigma_{rr}}{\shortparallel}} - \sigma_{zz} = Y \quad \text{at } r = 0$$

Yield $\quad \frac{\rho\omega^2}{32} 3D^2 + \frac{\nu\rho\omega^2}{32} D^2 = Y$

$$\omega^2 = \frac{32Y}{\rho D^2(3+\nu)}$$

$$\Rightarrow \omega = 1865.1 \text{ rad/s}$$

Displacement u at $r = \frac{D}{2}$

$$u\Big|_{r=\frac{D}{2}} = \frac{\rho\omega^2 D^3}{32E}\left[\frac{3}{2} - \frac{4}{8} + \nu(-1) - \nu^2\left(\frac{1}{2} - \frac{1}{2}\right)\right]$$

$$= \frac{\rho\omega^2 D^3}{32E}(1-\nu)$$

For disc to touch casing $w_c = 0.005\,m$

$$\Rightarrow \quad \omega^2 = \frac{32E}{\rho D^3(1-\nu)} w_c$$

$$\omega = 9780.8 \text{ rad/s}$$

ie yielding occurs first

2

(a) Must satisfy $\nabla^4 \phi = 0$

$$\frac{\partial^2 \phi}{\partial x^2} = 0 \; ; \quad \frac{\partial^2 \phi}{\partial y^2} = 2A + 6Cy + 6Dxy$$

$$\frac{\partial}{\partial x^2}\left(2A + 6Cy + 6Dxy\right) = 0$$

$$\frac{\partial}{\partial y^2}\left(2A + 6Cy + 6Dxy\right) = 0 \quad \Rightarrow \quad \nabla^4 \phi = 0$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2A + 6Dxy + 6Cy$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\sigma_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y} = -\frac{\partial}{\partial y}\left(By + Dy^3\right)$$

$$= -B - 3Dy^2$$

(b) Boundary conditions: $\sigma_{yy} = 0 \; @ \; y = \pm \frac{d}{2}$ ✓

& $\sigma_{xy} = 0 \; @ \; y = \pm \frac{d}{2}$

$$-B - \frac{3Dd^2}{4} = 0$$

$$B = -\frac{3Dd^2}{4}$$

$$\sigma_{xy} = -B\left(1 + \frac{3D}{B} y^2\right) = \frac{3D}{4}\left(d^2 - 4y^2\right)$$

$@ x=0 \qquad T = \int_{-\frac{d}{2}}^{d/2} \sigma_{xx}\, b\, dy$

$$= b\left[2Ay + 3Cy^2\right]_{-d/2}^{d/2}$$

$$= b\left[2Ad\right]$$

$$A = \frac{T}{2bd}$$

$$M = b\int_{-d/2}^{d/2} \sigma_{xx}\, y\, dy$$

$$= b\left[2Ay^2 + 2Cy^3\right]_{-d/2}^{d/2}$$

$$M = \frac{bCd^3}{2}$$

$$C = \frac{2M}{bd^3}$$

$$S = b\int_{-d/2}^{d/2} \sigma_{xy}\, dy$$

$$S = bD\left[\frac{3d^2y}{4} - y^3\right]_{-d/2}^{d/2}$$

$$= bD\left[\frac{3}{4}d^3 - \frac{d^3}{4}\right] = \frac{Dbd^3}{2} \Rightarrow D = \frac{2S}{bd^3}$$

(c)

For $M = T = 0$

$$\sigma_{xx} = \frac{12S}{bd^3} xy \quad ; \quad \sigma_{xy} = \frac{6S}{4bd^3}(d^2 - 4y^2)$$

$$= \frac{3}{2}\frac{S}{bd^3}(d^2 - 4y^2)$$

@ $x = l$ & $y = d/2$

$$\sigma_{xx} = \cancel{\frac{12S}{bd^3}} \frac{6S}{bd^2} l \quad ; \quad \sigma_{xy} = 0$$

$$Y = \frac{6Sl}{bd^2} \Rightarrow S = \frac{Ybd^2}{6l}$$

@ $x = l$ & $y = 0$

$$\sigma_{xx} = 0 \; ; \quad \sigma_{xy} = \frac{3S}{2bd} = \frac{Y}{2}$$

$$S = \frac{Ybd}{3}$$

For yield to occur simultaneously

$$\frac{d}{6l} = \frac{1}{3} \Rightarrow l = \frac{d}{2}$$

Result not meaningful as i) root close to loading

(ii) Displacement boundary conditions at $x = l$ most probably are not satisfied.

3

(a) $\nabla^2 \psi = -2G\beta$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{2m}{a^2} \;; \qquad \frac{\partial^2 \psi}{\partial y^2} = -\frac{2m}{b^2}$$

$$\nabla^2 \psi = \frac{-2m(a^2+b^2)}{a^2 b^2} = -2G\beta$$

$$\text{ie} \quad m = \frac{G\beta a^2 b^2}{a^2+b^2} \quad \checkmark$$

also must satisfy traction $= 0$ on boundary ie $\psi =$ constant

on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Yes with $\psi = 0$. ✓

(b) $T = 2\int_A \psi \, dA$

$$= 2m \int_A \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dA = \frac{2m\pi ab}{2}$$

$$T = \frac{G\beta a^2 b^2}{a^2+b^2} \pi a b$$

$$\beta = \frac{T}{G} \frac{a^2+b^2}{\pi a^3 b^3}$$

Max shear stress $x=0$, $y=b$ $\quad \frac{\partial \psi}{\partial y} = -\frac{2my}{b^2}\Big|_{y=b} = = -\frac{2m}{b}$

ie magnitude of shear stress $\frac{2m}{b} = \frac{2G\beta a^2 b^2}{a^2+b^2}$

$$= \frac{2T}{\pi a b^2}$$

(c) Consider $\psi = m\left(1 - \frac{y^2}{b^2}\right)$ over rectangle

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial y^2} = -\frac{2m}{b^2} = -2G\beta$$

$$m = G\beta b^2$$

$$T = 2\int \psi \, dA = 2m \; 2a \; \frac{4b}{3} = \frac{16mab}{3}$$

$$T = \frac{16 G\beta a b^3}{3}$$

For ellipse $\frac{T}{\beta} = \frac{G\pi a^3 b^3}{a^2+b^2}$

$a >> b \quad = G\pi a b^3$

$\Rightarrow$ rectangle is $\frac{16}{3\pi} = 1.7$ stiffer

m same in both cases when $a >> b$

Peak stress $= 2m$ in both cases.

4)

(a) ~~□~~ Use Lame's equations

$$\sigma_{rr} = A - \frac{B}{r^2}, \qquad \sigma_{\theta\theta} = A + \frac{B}{r^2}$$

$r \to \infty \qquad \sigma_{rr} = A = T$

$r = a \qquad \sigma_{rr} = 0$

$$\Rightarrow \quad 0 = T - \frac{B}{a^2} \Rightarrow \quad B = Ta^2$$

$$\sigma_{rr} = T\left(1 - \frac{a^2}{r^2}\right), \qquad \sigma_{\theta\theta} = T\left(1 + \frac{a^2}{r^2}\right)$$

$$\sigma_{\theta\theta} > \sigma_{rr} > \sigma_{zz} = 0$$

At edge of ~~□~~ hole by Tresca $\sigma_{\theta\theta} - \sigma_{zz} = Y$

$$2T = Y$$

$$T = \frac{Y}{2}$$

(b) In elastic region $\sigma_{rr} = A_1 - \frac{B_1}{r^2}$

$$\sigma_{\theta\theta} = A_1 + \frac{B_1}{r^2}$$

$r \to \infty, \ \sigma_{rr} \to A_1 \quad \Rightarrow \quad A_1 = T$

In plastic region $\sigma_{\theta\theta} = Y$ (from a)

$$\frac{d}{dr}\left(r\,\sigma_{rr}\right) = \sigma_{\theta\theta} = Y$$

$$\sigma_{rr} = Y + \frac{D}{r}$$

at $r = 0$ $\quad \sigma_{rr} = 0 \quad \Rightarrow \quad 0 = Y + \frac{D}{a}$

$$\sigma_{rr} = Y\left(1 - \frac{a}{r}\right)$$

$$\sigma_{\theta\theta} = Y$$

at $r = c$

σ_{rr} : $\quad Y\left(1 - \frac{a}{c}\right) = T - \frac{B_1}{c^2} \qquad (1)$

$\sigma_{\theta\theta}$: $\quad Y = T + \frac{B_1}{c^2} \qquad (2)$

(1) + (2) $\Rightarrow \quad Y\left(2 - \frac{a}{c}\right) = 2T$

$$\frac{c}{a} = \frac{Y}{2(Y - T)}$$

& $\quad B_1 = \frac{Yac}{2}$

(c)

For $T = \frac{3}{4} Y$

$\frac{c}{a} = 2$, $B_1 = Ya^2$

In plastic zone: $a \leq r \leq 2a$

$\sigma_{rr} = 1 - \frac{a}{r}$; $\sigma_{\theta\theta} = Y$

In ~~$\not{}$~~ elastic zone $r \geq 2a$

$\frac{\sigma_{rr}}{Y} = 0.75 - \frac{a^2}{r^2}$; $\frac{\sigma_{\theta\theta}}{Y} = 0.75 + \frac{a^2}{r^2}$