Engineering Tripos IIA 2020/2021
3C7 Crib

1. Use Lamé solution for elastic case $\sigma_{rr} = A - \frac{B}{r^2}$; $\sigma_{\theta\theta} = A + \frac{B}{r^2}$

B.C. gives A, B

As $r \to \infty$, $\sigma_{rr} \to \sigma_0$ $\Rightarrow$ $A = \sigma_0$

At $r = a$, $\sigma_{rr} = 0$ $\Rightarrow$ $B = a^2 \sigma_0$

$$\sigma_{rr} = \sigma_0 \left(1 - \left(\frac{a}{r}\right)^2\right) \quad ; \quad \sigma_{\theta\theta} = \sigma_0 \left(1 + \left(\frac{a}{r}\right)^2\right)$$

a(i) For thin sheet, assume $\sigma_{33} = 0$

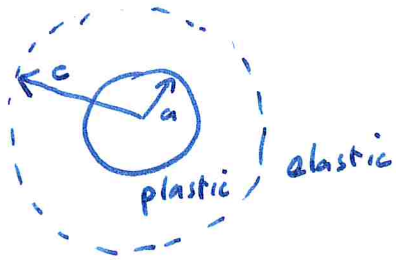
$\Rightarrow$ σ_{rr} is intermediate principal stress for elastic solution.

First yield at $r = a$, where $\sigma_{rr}, \sigma_{33} = 0$ (uniaxial stress)

$$\sigma_{\theta\theta} = 2\sigma_0 = Y$$

$$\sigma_0 = \frac{Y}{2}$$

(ii) For $\sigma^* > \frac{Y}{2}$



Assume σ_{rr} remains intermediate principal stress in plastic zone

$$\Rightarrow \sigma_{\theta\theta} - \sigma_{33} = \sigma_{\theta\theta} - 0 = Y$$

Equilibrium $\frac{d(r\sigma_{rr})}{dr} = \sigma_{\theta\theta} = Y$

$$r\sigma_{rr} = Yr + D$$

$$\sigma_{rr} = Y + \frac{D}{r}$$

a(ii) continued

Boundary condition, at $r = a$, $\sigma_{rr} = 0$ $\Rightarrow$ $D = -Ya$

$$\therefore \sigma_{rr} = Y\left(1 - \frac{a}{r}\right)$$

In elastic zone

$$\sigma_{rr} = A' - \frac{B'}{r^2} \quad ; \quad \sigma_{\theta\theta} = A' + \frac{B'}{r^2}$$

Boundary condition

at $r \to \infty$, $\sigma_{rr} \to \sigma^*$ $\Rightarrow$ $A = \sigma^*$

at $r = c$, $\sigma_{\theta\theta} = Y$, $\sigma_{rr} = Y\left(1 - \frac{a}{c}\right)$ to match elastic soln.

$$\therefore \sigma^* - \frac{B'}{c^2} = Y\left(1 - \frac{a}{c}\right) \quad ①$$

$$\sigma^* + \frac{B'}{c^2} = Y \quad ②$$

① + ②

$$2\sigma^* = Y\left(2 - \frac{a}{c}\right)$$

$$\therefore c = \left(\frac{Y}{\sigma^* - Y}\right) \cdot \frac{a}{2}$$

check: at $\sigma^* = Y$
$c \to \infty$ ✓
at $\sigma^* = \frac{Y}{2}$
$c = a$ ✓

(b) Assume plane strain $\Rightarrow$ for elastic solution σ_{33} is intermediate principal stress.

Assume this continues in plastic zone

$\Rightarrow$ yield condition becomes

$$\sigma_{\theta\theta} - \sigma_{rr} = Y$$

(b) continued

Rewrite equilibrium equation in plastic zone

$$\sigma_{\theta\theta} = \frac{d}{dr}(r\sigma_{rr}) = \sigma_{rr} + r\frac{d\sigma_{rr}}{dr}$$

$$\therefore r\frac{d\sigma_{rr}}{dr} = \sigma_{\theta\theta} - \sigma_{rr} = Y$$

$$\frac{d\sigma_{rr}}{dr} = \frac{Y}{r}$$

$$\sigma_{rr} = Y\ln\left(\frac{r}{f}\right)$$ for some constant f

B.C. at $r = a$, $\sigma_{rr} = 0$ $\Rightarrow$ $1 = \frac{a}{f}$ $\Rightarrow$ $f = a$

$$\therefore \sigma_{rr} = Y\ln\left(\frac{r}{a}\right);\ \sigma_{\theta\theta} = \sigma_{rr} + Y = Y\left[\ln\left(\frac{r}{a}\right)+1\right]$$

Match elastic soln. at $r = c$. As before $\sigma_{\theta\theta} = \sigma^* + \frac{B'}{r^2}$; $\sigma_{rr} = \sigma^* - \frac{B'}{r^2}$

$$\therefore \sigma_{\theta\theta}(c) = Y\left[\ln\left(\frac{c}{a}\right)+1\right] = \sigma^* + \frac{B'}{c^2} \quad (3)$$

$$\sigma_{rr}(c) = Y\ln\left(\frac{c}{a}\right) = \sigma^* - \frac{B'}{c^2} \quad (4)$$

(3) + (4)

$$2\ln\left(\frac{c}{a}\right) + 1 = 2\sigma^*$$

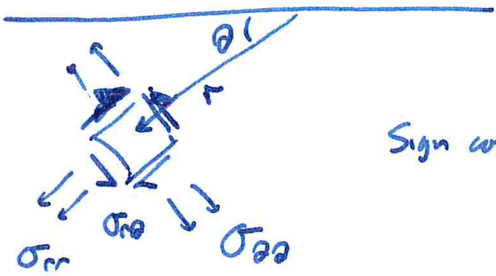
$$\ln\left(\frac{c}{a}\right) = \frac{\sigma^*}{Y} - \frac{1}{2}$$

$$\Rightarrow c = ae^{\left(\frac{\sigma^*}{Y} - \frac{1}{2}\right)}$$

check: at $\sigma^* = \frac{Y}{2}$, $c = a$ ✓

$\sigma^* = Y$, $c = ae^{\frac{1}{2}} = 1.65\,a$ - plausible.

2.



Sign convention.

(a) Use $\phi = Ar^2\theta$

$$\frac{\partial \phi}{\partial r} = 2Ar\theta \qquad \frac{\partial \phi}{\partial \theta} = Ar^2 \qquad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = A$$

$$\frac{\partial^2 \phi}{\partial r^2} = 2A\theta \qquad \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

(i) Check compatibility, $\nabla^4 \phi = 0$, for ϕ to be valid for <u>elastic</u> system.

$$\nabla^4 \phi = \left\{ \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right\} \psi$$

where $\psi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta^2} = 2A\theta + 2A\theta + 0$

$= 4A\theta$

$\therefore \nabla^2 \psi = \nabla^4 \phi = 0$ ✓

(ii) $\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} = 2A\theta$; $\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 2A\theta$; $\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -A$

For plane $y = 0$, either

$\theta = 0 \Rightarrow \sigma_{\theta\theta} = 0, \ \sigma_{r\theta} = -A$

$\theta = \pi \Rightarrow \sigma_{\theta\theta} = 2A\pi, \ \sigma_{r\theta} = -A$

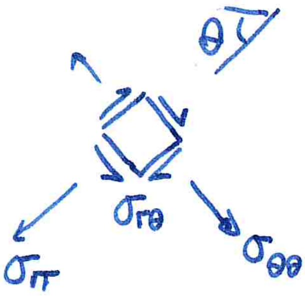
Matching tractions shown on figure.

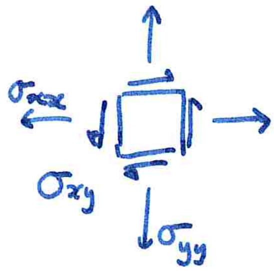
(a) (iii)

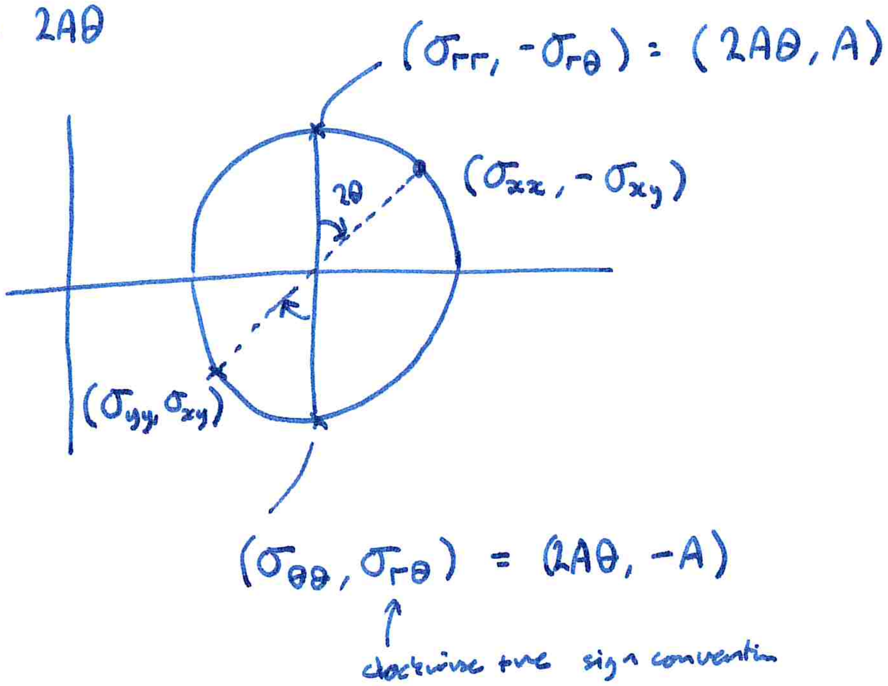
For a general position, plot Mohrs $\odot$

radius A

centred at $2A\theta$





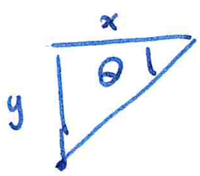


$$\therefore \quad \sigma_{xx} = 2A\theta + A\sin(2\theta)$$

$$\sigma_{yy} = 2A\theta - A\sin(2\theta)$$

$$\sigma_{xy} = -A\cos 2\theta$$

In general posn.



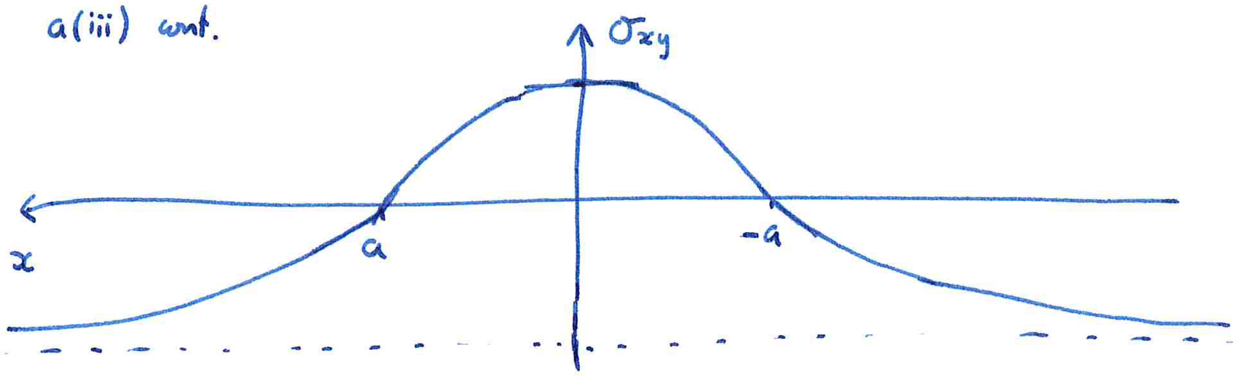
$$\cos\theta = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = \frac{x^2-y^2}{x^2+y^2}$$

At $y=a$

$$\therefore \quad \sigma_{xy} = \frac{A(a^2-x^2)}{a^2+x^2}$$

a(iii) cont.



(b) To match boundary traction, superpose.

(i) $\phi_1 = A r_1^2 \theta_1$ with origin at $x_1 = -a \Rightarrow x_1 = x + a$

(ii) $\phi_2 = -A r_2^2 \theta_2$ with origin at $x_2 = +a \Rightarrow x_2 = x - a$

and $A = \dfrac{p}{2\pi}$

gives $$\sigma_{xy1} = \frac{p(a^2 - x_1^2)}{2\pi(a^2 + x_1^2)}$$

$$= \frac{p}{2\pi} \frac{(a^2 - x^2 - 2ax - a^2)}{a^2 + x^2 + 2ax + a^2} = -\frac{px(x+2a)}{2\pi(x^2 + 2ax + 2a^2)}$$

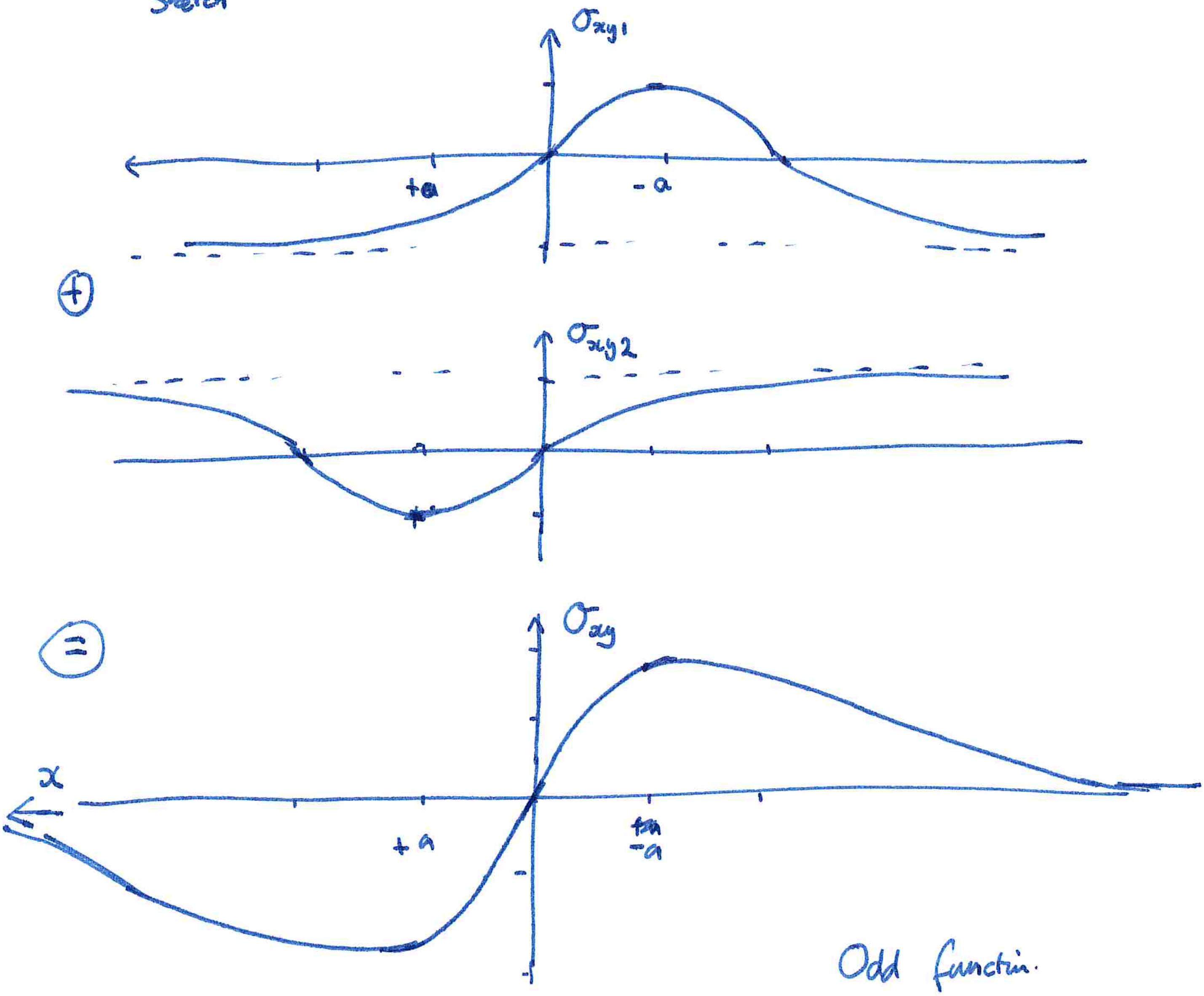
$$\sigma_{xy2} = \frac{-p(a^2 - x_2^2)}{2\pi(a^2 + x_2^2)}$$

$$= -\frac{p}{\pi} \frac{(a^2 - x^2 + 2ax \mp a^2)}{a^2 + x^2 - 2ax + a^2} = \frac{px(x - 2a)}{2\pi(x^2 - 2ax + 2a^2)}$$

$\Rightarrow \sigma_{xy} = \sigma_{xy1} + \sigma_{xy2}$

b (cont)

Sketch



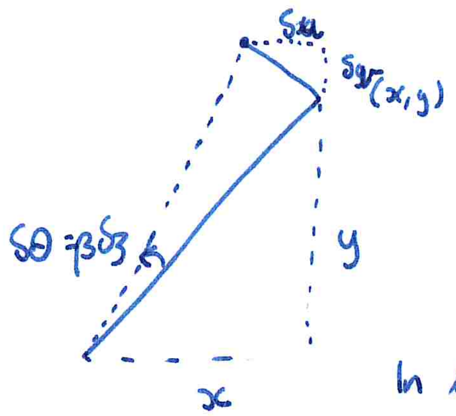
Odd function.

$\rightarrow 0$ at $x = \pm\infty$

Max at x slightly less than $-a$

Min at x slightly more than a.

3(a) Consider a point at $(x, y, \delta z)$ rotating around origin by $\delta\theta = \beta\,\delta z$



$$\delta u = -y\,\delta\theta = -\beta y\,\delta z$$
$$\delta v = x\,\delta\theta = +\beta x\,\delta z$$

In limit as $\delta z \to 0$

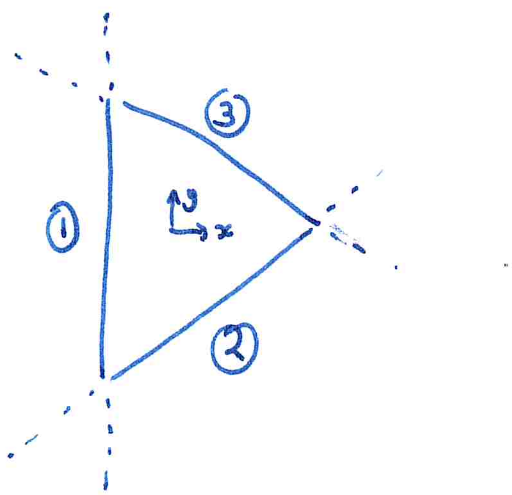
$$\frac{\partial u}{\partial z} = -\beta y \,;\quad \frac{\partial u}{\partial x} = \beta x$$

b(i) Require ψ = constant on boundary.

① At $x = -\frac{a}{3}$, $\left[\frac{x}{a} + \frac{1}{3}\right] = 0 \Rightarrow \psi = 0$

② At $x = \sqrt{3}y + \frac{2a}{3}$, $\left[\frac{x}{a} - \frac{\sqrt{3}y}{a} - \frac{2}{3}\right] = 0 \Rightarrow \psi = 0$

③ At $x = -\sqrt{3}y + \frac{2a}{3}$, $\left[\frac{x}{a} + \frac{\sqrt{3}y}{a} - \frac{2}{3}\right] = 0 \Rightarrow \psi = 0$



3(b)(ii) $$\psi = \frac{G\beta a^2}{2}\left[\frac{x}{a} - \frac{\sqrt{3}y}{a} - \frac{2}{3}\right]\left[\frac{x}{a} + \frac{\sqrt{3}y}{a} - \frac{2}{3}\right]\left[\frac{x}{a} + \frac{1}{3}\right]$$

$$\sigma_{3x} = \frac{\partial\psi}{\partial y} = \frac{G\beta a\sqrt{3}}{2}\left[\frac{x}{a} - \frac{\sqrt{3}y}{a} - \frac{2}{3}\right]\left[\frac{x}{a} + \frac{1}{3}\right] - \frac{G\beta a\sqrt{3}}{2}\left[\frac{x}{a} + \frac{\sqrt{3}y}{a} - \frac{2}{3}\right]\left[\frac{x}{a} + \frac{1}{3}\right]$$

$$= -G\beta a\left[\frac{y}{a}\right]\left[3\frac{x}{a} + 1\right]$$

$$\sigma_{3y} = -\frac{\partial\psi}{\partial x} = -\frac{G\beta a}{2}\left[\frac{x^2}{a^2} - \frac{4x}{3a} + \frac{4}{9} - \frac{3y^2}{a^2}\right] - \frac{G\beta a}{2}\left[\frac{x}{a} + \frac{1}{3}\right]\left[\frac{2x}{a} - \frac{4}{3}\right]$$

$$= -G\beta a\left[\frac{3}{2}\frac{x^2}{a^2} - \frac{x}{a} - \frac{3y^2}{2a^2}\right]$$

For elastic soln., require $\cancel{\gamma}$ $\nabla^2\psi = -2G\beta$

$$\frac{\partial^2\psi}{\partial x^2} = G\beta\left(3\frac{x}{a} - 1\right)$$

$$\frac{\partial^2\psi}{\partial y^2} = G\beta\left(-3\frac{x}{a} - 1\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2G\beta \checkmark$$

3(c) From shear strain expressions, with $\sigma_{zx} = G\gamma_{zx}$

$\sigma_{zy} = G\gamma_{zy}$

$$\frac{\partial w}{\partial x} = -\frac{\partial u}{\partial z} + \gamma_{zx} = \beta a . \frac{y}{a} - \beta a \left[\frac{y}{a}\right]\left[\frac{3x}{a} + 1\right]$$

$$= -3\beta a \left(\frac{xy}{a^2}\right)$$

$$\therefore w = -\frac{3}{2}\beta a^2 \left(\frac{x^2 y}{a^3}\right) + f(y) \qquad (1)$$

$$\frac{\partial w}{\partial y} = -\frac{\partial v}{\partial z} + \gamma_{zy} = -\beta a . \frac{x}{a} - \beta a \left[\frac{3}{2}\frac{x^2}{a^2} - \frac{x}{a} - \frac{3y^2}{a^2}\right]$$

$$= -\beta a \left[\frac{3}{2}\frac{x^2}{a^2} - \frac{3y^2}{2a^2}\right]$$

$$\therefore w = -\frac{3}{2}\beta a^2 \left(\frac{x^2 y}{a^3}\right) + \frac{\beta a^2}{2}\frac{y^3}{a^3} + g(x) \qquad (2)$$

From (1) & (2) $f(y) = \frac{\beta a^2 y^3}{2a^3} + C$, $g(x) = C$ where C is displacement of origin.

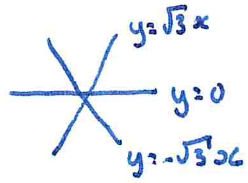
Choose $C = 0$ gives

$$w = -\frac{3}{2}\beta a^2 \left(\frac{x^2 y}{a^3}\right) + \frac{\beta a^2}{2}\frac{y^3}{a^3}$$

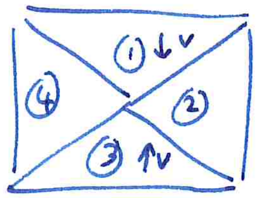
check: $w = 0$ at $y = 0$

or $y^* = \pm\sqrt{3}\,x$

i.e. $w = 0$ on symmetry lines ✓



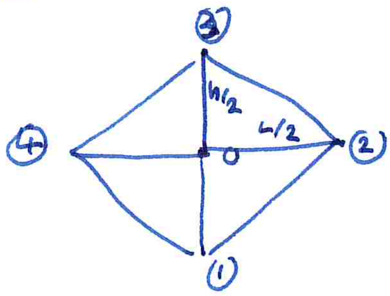
4.(a) Velocities



$$l_{12}^2 = \frac{h^2+L^2}{4}$$

Velocity diagram

$\left(\frac{2v}{h}\right)\times$



Rate of work done $= 2Fv$

Rate of energy dissipated $= k v_{12} l_{12} \times 4$ (symmetry)

$$= k \cdot \frac{2v}{h} \cdot l_{12}^2 \times 4$$

$$= 2kv\frac{(h^2+L^2)}{h}$$

Equate & cancel v

$$\Rightarrow F = k\frac{(h^2+L^2)}{h}$$

or $\dfrac{F}{(kL)} = \left(\dfrac{h}{L} + \dfrac{L}{h}\right)$ in dimensionless quantities.

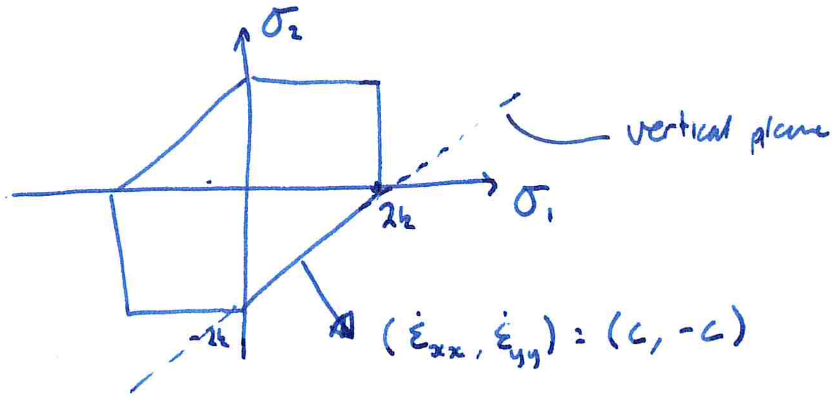
b(i) $\dot{\varepsilon}_{xx} = \frac{\partial \dot{u}}{\partial x} = c$

$\dot{\varepsilon}_{yy} = \frac{\partial \dot{v}}{\partial y} = -c$

note - volume conserv.

& $\dot{\gamma}_{xy} = 0$, i.e. these are principle stem..

~~Cons~~ Draw Tresca for $\sigma_3 = 0$



~~Re~~ Normality implies $(\sigma_1 - \sigma_2) = 2k$ is yield criterion, and rate of energy dissipated $\dot{\omega}$

$$\dot{\omega} = \dot{\varepsilon}_1 \sigma_1 + \dot{\varepsilon}_2 \sigma_2$$
$$= 2kc \quad \text{whatever stress state.}$$

(ii) Energy dissipated : In volume $W_v = 2kc \times Lh$

Along slip lines $y = \pm \frac{h}{2}$.

Consider $x > 0$, $y = h/2$

Relative velocity between die & material $= \dot{u} = cx$

Total dissipated for this slip line $= \int_0^{L/2} cx\,k\,dx = \left[kc\frac{x^2}{2}\right]_0^{L/2} = \frac{kcL^2}{8}$

3b(ii) cont.

$x>0$, $x<0$ for $h = \pm L/2$

$$\dot{W} = 2kc \times Lh + 4 \times \frac{kcL^2}{8}$$

$$= 2kc\,Lh + \frac{kcL^2}{2}$$

Rate of Work done $= \frac{Fch}{2} + \frac{Fch}{2}$ (top & bottom)

$$\therefore Fch = 2kc\left(Lh + \frac{L^2}{4}\right)$$

$$\frac{F}{kL} = 2 + \frac{1}{2}\left(\frac{L}{h}\right)$$

ENGINEERING TRIPOS PART IIA 2021

MODULE 3C7: Mechanics of Solids

The examination was taken by 53 candidates for Part IIA. The raw marks had an average of 55.2% with the best at 93% and the worst at 18.7%.

Q1 Plasticity of a metal sheet with a circular hole in the centre

47 attempts, Average mark 48.7/100, Maximum 100, Minimum 5.

A popular question. Most students answered question (a) well, but most students confused plane strain and plane stress conditions, leading to part a(ii) and (b) being switched. Most students did not know what to do in question (b) as a result.

Q2 Plane-stress elastic response of a semi-infinite plate under pressure

45 attempts, Average mark 49.6/100, Maximum 97, Minimum 28.

A popular question, for which part (a) is straightforward. Most students did not use the correct angle on the Mohr's circle of stress to transform the stresses from polar to Cartesian coordinates. Few students were then able to do the superposition in question (b).

Q3 Axial torsion of an elastic shaft

49 attempts, Average mark 63.5/100, Maximum 98, Minimum 13.

A popular and straightforward question that required thorough algebra. Most students answered the question well to part (b) and (c), albeit algebraic mistakes. Almost no students could understand how to use a FBD at small displacement to answer question (a).

Q4 Plastic response of a rectangular rod under plane stress compression

17 attempts, Average mark 64.9/100, Maximum 97, Minimum 29.

This question was less popular than the others, but those who answered it generally did well, in particular with question (a) and (b)(i). The main difficulty with this question was to use a distributed field to find the upper bound, however, most students who understood what to do and attempted it answered the question well.

Dr Christelle Abadie (Principal Assessor)