Version VSD/2

EGT2
ENGINEERING TRIPOS PART IIA

Thursday 1 May 2014 9.30 to 11

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C7 datasheet (2 pages).
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) A thin flywheel of uniform thickness, with an outer diameter D and no central hole, is made of material with density ρ, Young's modulus E and Poisson's ratio ν. When the disk is spinning with rotational speed ω, the elastic stress distribution is given as a function of radius r by

$$\sigma_{rr} = \frac{\rho\omega^2}{32}(3D^2 - 12r^2) + \frac{\nu\rho\omega^2}{32}(D^2 - 4r^2)$$
$$\sigma_{\theta\theta} = \frac{\rho\omega^2}{32}(3D^2 - 4r^2) + \frac{\nu\rho\omega^2}{32}(D^2 - 12r^2)$$

Show that this distribution satisfies equilibrium, compatibility, and relevant boundary conditions. [50%]

(b) A thin flywheel is made of steel with a uniaxial yield stress of 450 MPa, $E = 210$ GPa, $\nu = 0.3$ and $\rho = 7840$ kg m^{-3}. The flywheel has a diameter $D = 0.4$ m and no central hole and is spinning within a case that has an inner diameter 0.401 m. Calculate the rotational speed at which the flywheel will first fail, either by first yielding (assuming a Tresca yield criterion), or by binding on its case. [50%]

2 The Airy Stress Function ϕ

$$\phi = Ay^2 + Bxy + Cy^3 + Dxy^3$$

has been suggested as a suitable way of finding the stresses in the cantilever shown in Fig. 1, of length l, depth d and breadth b, subject to end loading with a tension T, bending moment M and shear force S.

(a) Show that ϕ is a suitable stress function for an elastic problem, and calculate the corresponding stresses. [20%]

(b) Find the constants A, B, C, D so that the stresses satisfy the boundary conditions of the problem. [50%]

(c) For the case $M = 0$, $T = 0$, use the solution for the stresses to calculate the geometry of the cantilever for which yield will occur simultaneously at $y = 0, x = l$ and $y = d/2, x = l$ using the Tresca yield criterion. Comment on the likely practical accuracy of this result. [30%]

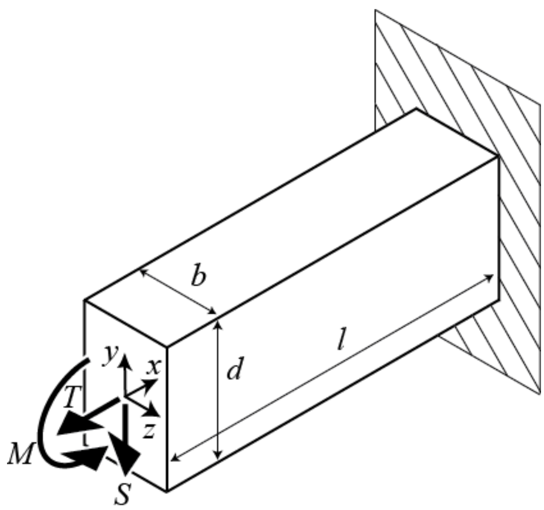

Fig. 1

3 Figure 2 shows the elliptical cross-section of a bar with maximum and minimum cross-sectional dimensions $2a$ and $2b$ respectively, made of a material with shear modulus G. The Prandtl stress function

$$\psi = m\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

is to be used to explore the elastic torsional response of the bar, where m is a constant.

(a) Show that ψ is a suitable Prandtl stress function for this problem. [20%]

(b) For an applied torque T, find the resultant twist/unit length of the bar. What is the value of the maximum shear stress within the bar in terms of the applied T? (Note that the integral over the area of the ellipse, $\iint (1 - x^2/a^2 - y^2/b^2)\, dA = \pi ab/2$.) [50%]

(c) For $a \gg b$, for a given twist/unit length, compare the torque and peak shear stress carried by this section with the equivalent results for a rectangular section of dimension $2a \times 2b$. [30%]

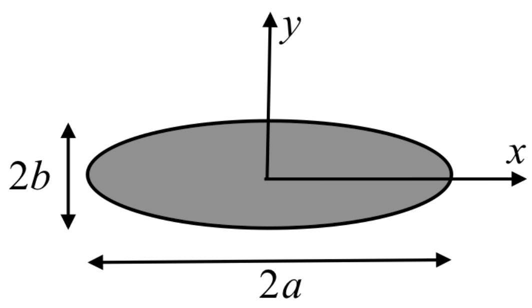

Fig. 2

4 An infinitely large thin sheet containing a circular hole of radius a is subjected to a remote equi-biaxial stress $\sigma_{rr} = T$ as shown in Fig. 3. The sheet is made from an elastic perfectly-plastic material with a uniaxial tensile strength Y and yields according to the Tresca yield criterion.

(a) Show that first yield occurs at $T = Y/2$ and that the minimum principal stress is zero throughout the plate. [30%]

(b) The applied stress is then increased to $T > Y/2$ and a circular plastic zone of radius c extends concentrically from the edge of the hole. Determine an expression for c in terms of the applied stress T. [50%]

(c) Sketch the distribution of the stresses σ_{rr} and $\sigma_{\theta\theta}$ for $r \geq a$ when a stress $T = 0.75Y$ is applied. [20%]

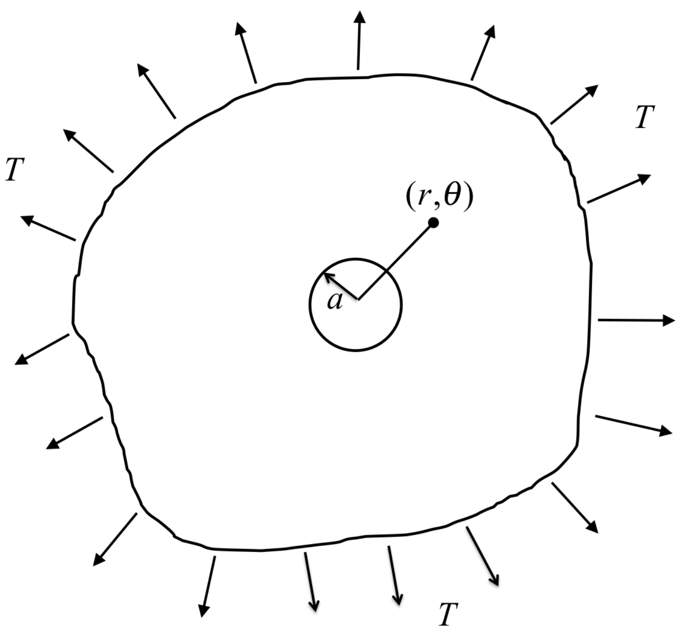

Fig. 3

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Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{\mathrm{d}(r\sigma_{rr})}{\mathrm{d}r} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{\mathrm{d}(r^2\sigma_{rr})}{\mathrm{d}r}$
Lamé's equations (in elasticity)	$\sigma_{rr} = \mathrm{A} - \dfrac{\mathrm{B}}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\mathrm{d}r$	$\sigma_{rr} = \mathrm{A} - \dfrac{\mathrm{B}}{r^3}$
	$\sigma_{\theta\theta} = \mathrm{A} + \dfrac{\mathrm{B}}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\mathrm{d}r - E\alpha T$	$\sigma_{\theta\theta} = \mathrm{A} + \dfrac{\mathrm{B}}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \overline{E} = \dfrac{E}{1-\nu^2}\ ;\ \overline{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \overline{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (=\tau_x) = \dfrac{\partial \psi}{\partial y}, \quad \sigma_{zy}\ (=\tau_y) = -\dfrac{\partial \psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi\, dA$

Governing equation for elastic torsion: $\nabla^2\psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\prod = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, \mathrm{d}V$, $W = \underset{\sim}{P}^{\,T} \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P{}^3 - I_1\, \sigma_P{}^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\,\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right\}^{1/2}$

Equivalent strain increment $\mathrm{d}\bar{\varepsilon} = \sqrt{\frac{2}{3}}\,\left\{\mathrm{d}\varepsilon_1{}^2 + \mathrm{d}\varepsilon_2{}^2 + \mathrm{d}\varepsilon_3{}^2\right\}^{1/2}$

7. Yield criteria and flow rules

<u>Tresca</u>

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $\mathrm{d}\varepsilon_1 : \mathrm{d}\varepsilon_2 : \mathrm{d}\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

<u>von Mises</u>

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{\mathrm{d}\varepsilon_1}{\sigma'_1} = \frac{\mathrm{d}\varepsilon_2}{\sigma'_2} = \frac{\mathrm{d}\varepsilon_3}{\sigma'_3} = \frac{\mathrm{d}\varepsilon_1-\mathrm{d}\varepsilon_2}{\sigma_1-\sigma_2} = \frac{\mathrm{d}\varepsilon_2-\mathrm{d}\varepsilon_3}{\sigma_2-\sigma_3} = \frac{\mathrm{d}\varepsilon_3-\mathrm{d}\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{\mathrm{d}\bar{\varepsilon}}{\bar{\sigma}}.$$

MODULE 3C7 (Mechanics of Solids): Numerical answers to Tripos examination 2014

1. (b) Yielding occurs first

2. (c) for yield to occur simultaneously $l = d/2$.

3. (b) $\tau_{max} = (2T)/(\pi ab^2)$
   (c) rectangle is 1.7 times stiffer

4. (b) $\frac{c}{a} = Y/[2(Y - T)]$