EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 27 April 2021 9.00 to 10.40

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Attachment: 3C7 formulae sheet (2 pages).
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 An infinite sheet with a circular hole of radius a is loaded by an axisymmetric remote tensile stress σ_0. The sheet material is elastic-perfectly plastic with a tensile yield strength Y, and obeys the Tresca yield criterion.

(a) Assume that the sheet thickness is much less than a.

(i) Calculate the value of σ_0 at which yielding is initiated in the plate. [20%]

(ii) If the plate is loaded by a stress σ^* which is greater than the critical value to initiate yield, calculate the radius c of the plastic zone. [40%]

(b) If the thickness of the plate is much greater than a, calculate the radius c of the plastic zone in terms of the remote stress σ^*. [40%]

2 The superposition of two Airy stress functions is to be used to analyse the plane stress elastic response of a semi-infinite plate to pressure loading on a patch, as sketched in Fig. 1(b).

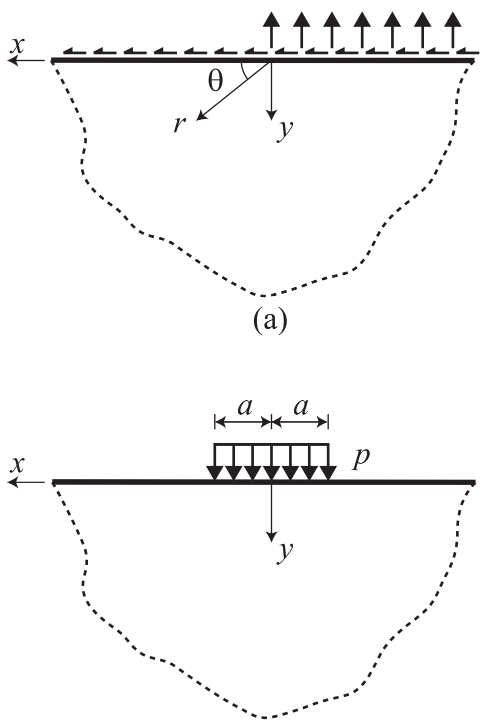
(a) Consider initially the Airy stress function $\phi = Ar^2\theta$ for the plate shown in Fig. 1(a).

(i) Show that ϕ is a valid Airy stress function for an elastic system. [10%]

(ii) In terms of A, calculate the stress components derived from ϕ in polar co-ordinates. Calculate the magnitude of the equilibrium tractions applied on the boundary plane $y = 0$, as sketched in Fig. 1(a). [20%]

(iii) Find σ_{xx}, σ_{yy}, σ_{xy} in the plate as a function of r and θ. Sketch a graph of the variation of σ_{xy} with x on the plane $y = a$. [30%]

(b) Using superposition of stress functions with an origin at $x = +a$ and $x = -a$, or otherwise, calculate σ_{xy} on the plane $y = a$ when the plate is loaded by a normal pressure p over the region $-a \leq x \leq a$, as shown in Fig. 1(b). Sketch a graph of the variation of σ_{xy} with x on the plane $y = a$. [40%]



(a)

(b)

Fig. 1

3 Consider axial torsion of an elastic shaft with the equilateral triangular cross section shown in Fig. 2.

(a) If β is the twist per unit length of the shaft, show that the in-plane displacements u and v in the x and y directions respectively obey

$$\frac{\partial u}{\partial z} = -\beta y \quad ; \quad \frac{\partial v}{\partial z} = \beta x. \qquad [10\%]$$

(b) The stresses in the shaft can be derived from the Prandtl stress function

$$\psi = \frac{G\beta a^2}{2}\left[\frac{x}{a} - \frac{\sqrt{3}y}{a} - \frac{2}{3}\right]\left[\frac{x}{a} + \frac{\sqrt{3}y}{a} - \frac{2}{3}\right]\left[\frac{x}{a} + \frac{1}{3}\right]$$

where $2a/\sqrt{3}$ is the side-length of the triangular cross-section, G is the shear modulus of the material and β is the twist per unit length of the shaft.

(i) Show that ψ satisfies the necessary boundary conditions. [10%]

(ii) Calculate the stresses σ_{yz} and σ_{xz} in the shaft, and show that these are an elastic solution. [40%]

(c) The shear strains in the shaft are given by

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad ; \quad \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

Calculate the out-of-plane displacement field w. [40%]

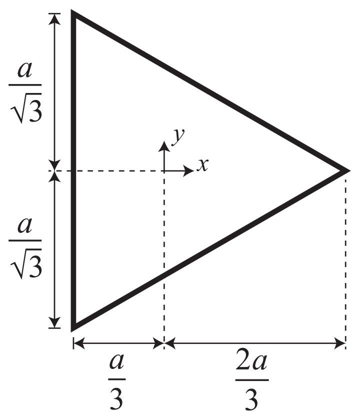


Fig. 2

4 A rod of elastic-perfectly plastic metal of rectangular cross-section is compressed in plane strain between two rigid dies. The rod has thickness h and width L, as shown in cross-section in Fig. 3(a). The force applied per unit length along the rod is F. The metal is isotropic and incompressible, with a flow stress in shear of magnitude k. Friction prevents any slip between the dies and the metal.

(a) By considering the tangential velocity discontinuities shown by dashed lines in Fig. 3(b), calculate an upper bound for the value of F required to deform the rod. [30%]

(b) An alternative velocity field is given by

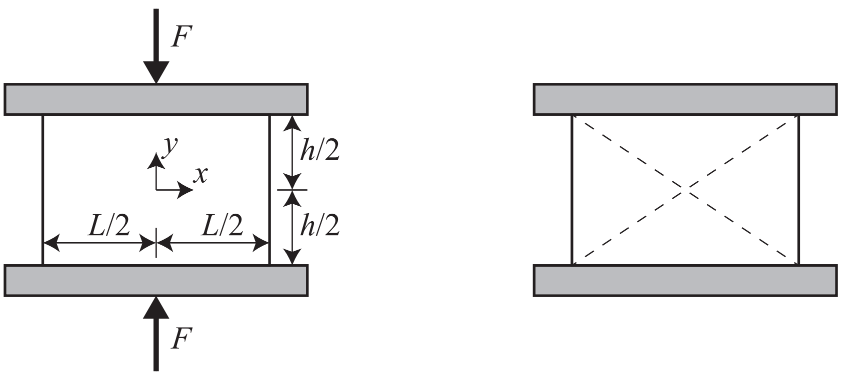
$$\dot{u} = cx$$
$$\dot{v} = -cy$$

where $\dot{u}$ and $\dot{v}$ are the velocities in the x and y directions respectively, and c is a constant.

(i) Show that the strain rate field is uniform. By assuming a Tresca yield criterion and considering the normality criterion, show that the rate of energy dissipated per unit volume is given by

$$\dot{w} = 2kc$$ [30%]

(ii) Use this strain rate field to find an alternative upper bound for F. Note that a velocity discontinuity is necessary at $y = \pm h/2$. [40%]



(a) (b)

Fig. 3

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Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \overline{E} = \dfrac{E}{1-\nu^2}\ ;\ \overline{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \overline{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial\theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial\theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity with no thermal strains or body forces)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{\partial \psi}{\partial y}, \quad \sigma_{zy}\ (= \tau_y) = -\dfrac{\partial \psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi\, dA$

Governing equation for elastic torsion: $\nabla^2 \psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1 \sigma_P^2 + I_2 \sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}} \left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}} \left\{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\right\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$