EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 24 April 2018 9.30 to 11.10

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C7 formulae sheet (2 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 A uniform bar of elliptical cross section, with semi-major axis a and semi-minor axis b, is made from a linear elastic solid of shear modulus G. A Cartesian co-ordinate system (x, y) is introduced in the bar cross-section such that the surface of the bar satisfies the relation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

The Prandtl stress function

$$\psi(x, y) = A + Bx^2 + Cy^2$$

is a candidate for determining the torsional response of the bar, where A, B and C are constants.

(a) Show that ψ is a suitable stress function for this problem. [30%]

(b) Determine the torque T in terms of G, a, b and the twist per unit length, β, and thereby determine the shear stress distribution along the minor axis of the ellipse. [50%]

Note: You may make use of the below integral over the ellipse:

$$\iint \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \mathrm{d}A = \frac{\pi ab}{2}.$$

(c) Make use of the above results to determine the torsional rigidity T/β for the elliptical bar when it contains a central elliptical hole of semi-major axis $a/2$ and semi-minor axis $b/2$. [20%]

2 (a) Show that

$$\phi(r,\theta) = C_1 \sin 2\theta + C_2\theta + C_3 \cos 2\theta \quad (1)$$

is a valid Airy stress function in polar coordinates (r,θ). [20%]

(b) Figure 1 is a side view of a tapered bean, of unit width w into the plane of the paper, that is loaded at each end by a moment M. It has been proposed that the stress function in eq. (1) can be used to determine the stress field in the beam.

(i) Write down the stress boundary conditions to be satisfied on the top and bottom faces. [10%]

(ii) Show that stress derived from $\phi(r,\theta)$ satisfies these boundary conditions and hence determine the relation between C_1 and C_2, and the value of C_3. [25%]

(c) For the applied moment M, determine the stresses in the beam as a function of r and θ. [20%]

(d) Show that for a long thin beam (where $\alpha \ll 1$), the stresses σ_{rr} are those that would be expected from simple beam theory. [25%]

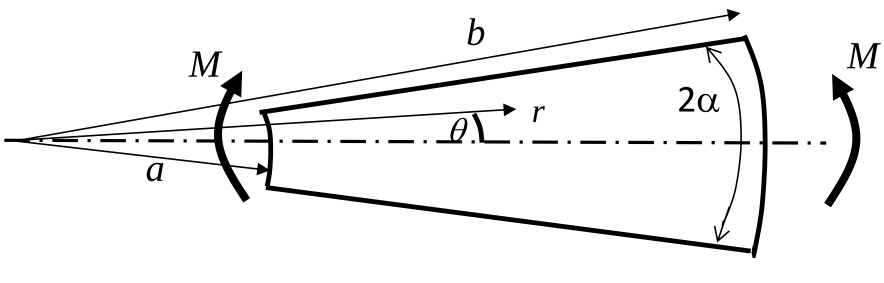

Fig. 1

3 (a) Discuss the conditions under which the in-plane stress state in a thin circular disk made from an elastic solid has circular symmetry. [15%]

(b) Starting from the general expressions of equilibrium provided on the datasheet, show that for a thin circular disk with circular symmetry, the equilibrium equation expressed in polar coordinates (r,θ) is given by

$$r\frac{\partial \sigma_{rr}}{\partial r} = \sigma_{\theta\theta} - \sigma_{rr},$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial and hoop stresses, respectively. [15%]

(c) Starting from the general compatibility equations provided on the datasheet, show that for the circular disk described in (b), the compatibility equations reduce to

$$r\frac{\partial \varepsilon_{\theta\theta}}{\partial r} = \varepsilon_{rr} - \varepsilon_{\theta\theta},$$

where ε_{rr} and $\varepsilon_{\theta\theta}$ are the radial and hoop strains, respectively. [25%]

(d) Consider a thin circular plate with a central hole of radius a. The plate is made of material with Young's modulus E, Poisson's ratio ν and coefficient of thermal expansion α. The plate is initially stress-free at a uniform temperature T_0. It is then heated axisymmetrically to a temperature $T(r)$. Derive the stress versus strain relationship for the heated plate in polar coordinates.

[15%]

(e) With u denoting the radial displacement, the equilibrium equation for the hollow circular plate described in (d) is expressed as:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (ur)}{\partial r}\right] = (1+\nu)\,\alpha\frac{\partial (T-T_0)}{\partial r}.$$

Solve for the displacements in the heated plate. [30%]

4 (a) A thin-walled tube is subjected to combined uniform axial tension and torsion. The axial stress is $\sigma = Y/2$ where Y is the tensile yield strength. The Tresca yield criterion is assumed. Find the magnitude of the shear stress τ on the cross section due to the applied torque at which the tube begins to yield, and determine the corresponding principal strain increments. [20%]

(b) A long cylindrical hole with internal radius a is bored vertically into the ground and subjected to an internal pressure of p. The ground can be approximated as a homogeneous elastic perfectly-plastic material of tensile yield strength Y and yields according to the Tresca criterion. The deformation of the ground is to be assumed as plane strain in the depth direction z.

(i) Determine the stresses σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} for an elastic response when first yield occurs. [40%]

(ii) When the internal pressure is increased, a circular plastic zone of radius c extends concentrically from the edge of the hole. Determine the relationship between the internal pressure p and the plastic zone radius c. [40%]

END OF PAPER

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Module 3C7: Mechanics of Solids
ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \bar{E} = \dfrac{E}{1-\nu^2}\ ;\ \bar{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \bar{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity with no thermal strains or body forces)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{\partial \psi}{\partial y}, \quad \sigma_{zy}\ (= \tau_y) = -\dfrac{\partial \psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi \, dA$

Governing equation for elastic torsion: $\nabla^2 \psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^{\,T} \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1 \sigma_P^2 + I_2 \sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}} \left\{ d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2 \right\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}} .$$