Engineering Tripos Part IIA

Module 3C8 Machine Design

Solutions 2018

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Assessor's comments:

Q1 Hertz contact in thrust bearing

A straightforward and popular question. The most common errors were: in (b) to forget that the approach of the plates is twice the displacement at a single contact; in (c) to assume equal force distribution between the three balls; and in (d) to consider only deflection of the larger ball.

Q2 Spin velocities and forces in angular contact ball bearing

This was the least popular question despite its similarity to an examples paper question. The most common error was not to impose an angular velocity on the assembly before calculating the spin velocities. There were also many sign errors, and answers that were dimensionally inconsistent.

Q3 Epicyclic gears

In answering (a) several candidates unnecessarily introduced torques and the virtual power equation. In parts (b) and (c) many candidates attempted unsuccessfully to use a tabular method of solution, despite the pointer in part (a) to use an algebraic method. In part (b) the torque was often incorrectly assumed to be divided equally between the two motors. In part (c) the brake torque on the carrier was often neglected.

Q4 Rack and pinion

Part (b)(i) required calculation of the minimum number of teeth to avoid interference. Many candidates had only a vague idea of how to do this, and there were few correct answers. The calculation of pinion torque and contact pressure in (b)(i) was more straightforward but there were many errors made in the contact force per unit width and the radius of curvature.

1. a) Conditions for Hertz analysis:

- smooth profiles – no sharp edges
- strains are small
- interfaces are frictionless
- behaviour is elastic
- surfaces are not close-fitting (conformal)

b) With 5 equal balls, $W = \frac{1}{5}\,kN = 200\,N$.

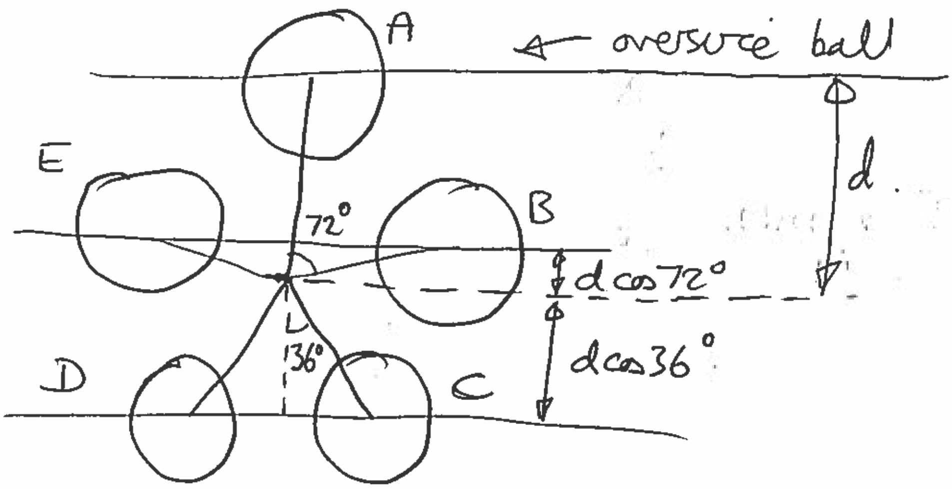
$E^* = 115\,GPa$, $R = 0.01\,m$

from data sheet
$$p_0 = \frac{1}{\pi}\left\{\frac{6\,P E^{*2}}{R^2}\right\}^{\frac{1}{3}}$$
$$= \frac{1}{\pi}\left\{\frac{6 \cdot 200N \cdot (115 \cdot 10^9\,N/m^2)^2}{(0.01\,m)^2}\right\}^{\frac{1}{3}}$$
$$= 1.72 \cdot 10^9\,N/m^2$$

approach of centres – data sheet
$$\delta = \frac{a^2}{R} = \frac{1}{2}\left\{\frac{9}{2}\,\frac{P^2}{E^{*2}\,R}\right\}^{\frac{1}{3}}$$
$$= \frac{1}{2}\left\{\frac{9}{2}\,\frac{(200N)^2}{(115 \cdot 10^9 N/m^2)^2\ 0.01\,m}\right\}^{\frac{1}{3}}$$
$$\delta = 5.54 \cdot 10^{-6}\,m$$

The plates will displace by 2δ relative to each other, $2 \times 5.54\,10^{-6}\,m = \underline{11.1\,\mu m}$

c)



assume ball A is oversize.
assume B and E not in contact.

take moments about CD.

$$P_A\left(d + d\cos 36^\circ\right) = W.\,d\cos 36^\circ$$

$$P_A = 1\,kN.\ \frac{\cos 36^\circ}{1+\cos 36^\circ}$$

$$\underline{P_A = 447\,N}$$

sum forces in vertical direction

$$P_A + P_C + P_D = W$$

$$\therefore \quad P_C + P_D = W - P_A = 1\,kN - 447\,N$$

but $P_C = P_D = \dfrac{1\,kN - 447\,N}{2} = 276.5\,N$

Hence $P_A = 447\,N$ is worst case

data sheet : $p_o = \dfrac{1}{\pi}\left\{\dfrac{6\,P_A E^{*2}}{R^2}\right\}^{\frac{1}{3}}$

$$= \frac{1}{\pi}\left\{\frac{6.\,447N.\,\left(115.10^9\,N/mm^2\right)^2}{(0.01001\,m)^2}\right\}^{\frac{1}{3}}$$

$$\underline{p_o = 2.25\,.10^9\,N/m^2}$$

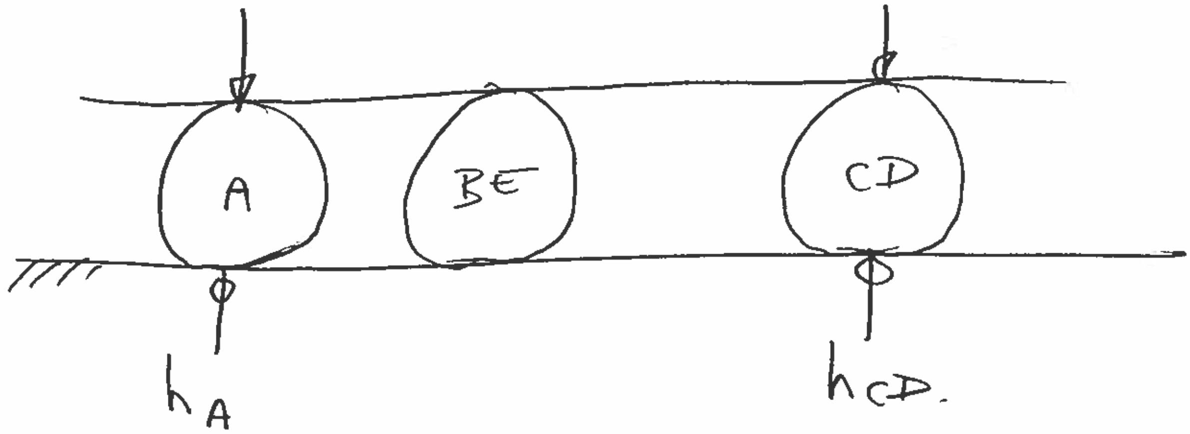
d) Find 2δ at A and at CD,

$$\delta \propto \left(\frac{P^2}{R}\right)^{\frac{1}{3}} \quad \therefore \quad \delta_A = 5.54 \cdot 10^{-6}\left(\left(\frac{2447}{200}\right)^2 \cdot \frac{10.00}{10.01}\right)^{\frac{1}{3}}$$

$$\delta_A = 9.467 \cdot 10^{-6}\,\text{m}.$$

and $$\delta_{CD} = 5.54 \cdot 10^{-6}\left(\frac{276.5}{200}\right)^{\frac{2}{3}}$$

$$\delta_{CD} = 6.875 \cdot 10^{-6}\,\text{m}$$

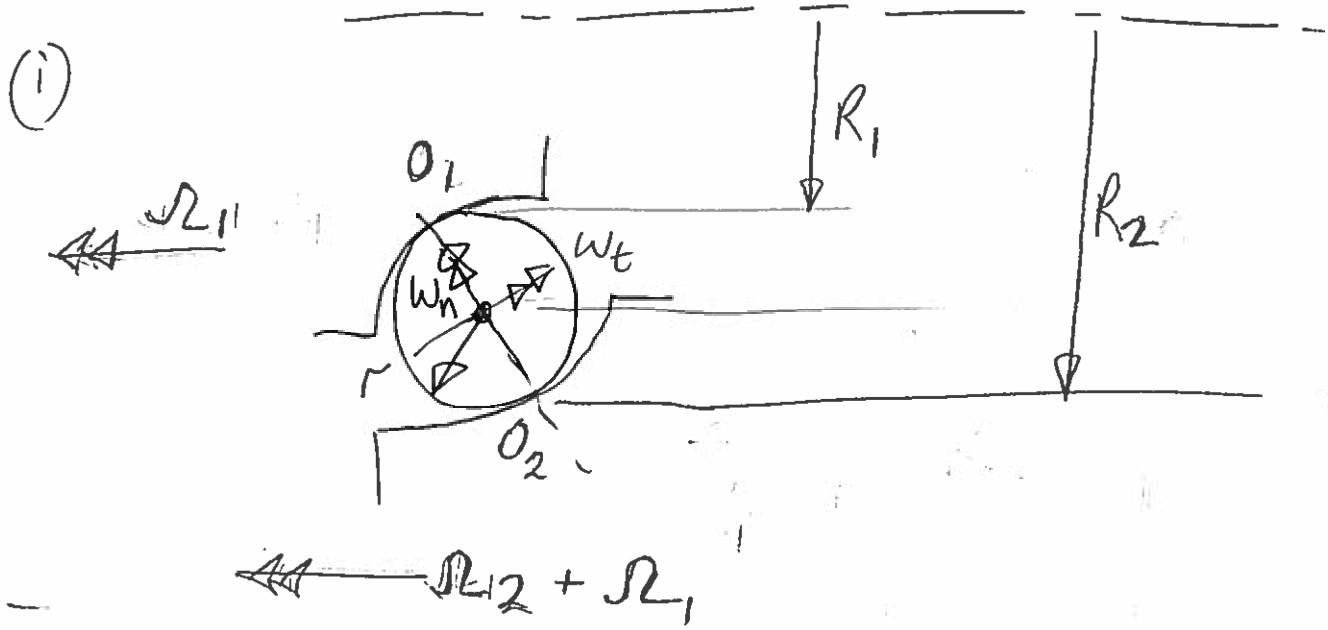


hence
$$h_A = 20.02\,\text{mm} - 2\delta_A$$
$$= 20.001\,\text{mm}$$

$$h_{CD} = 20.00\,\text{mm} - 2\delta_{CD}$$
$$= 19.986\,\text{mm}.$$

by inspection the distance between the plates at BE will be less than 20 mm, hence assumption of no contact is incorrect.

2(a) (i)



Add Ω_1 to whole assembly

Assume centre of ball is stationary and assume no sliding at O_1 and O_2. Consider velocities out of plane of paper

O_1 / $\qquad \Omega_1 R_1 = \omega_t r \qquad (1)$

O_2 / $\qquad (\Omega_2 + \Omega_1) R_2 = -\omega_t r \qquad (2)$

(1) + (2) $\qquad \Omega_1 R_1 + (\Omega_2 + \Omega_1) R_2 = 0$

$$\Omega_1 = -\Omega_2 \frac{R_2}{R_1 + R_2} \qquad (3)$$

subst (3) into (1)

$$\omega_t r = -\frac{R_1 R_2}{R_1 + R_2} \Omega_2$$

$$\omega_t = -\frac{R_1 R_2}{r(R_1 + R_2)} \Omega_2$$

ii) Spin at O_1

$$\Delta w_1 = \frac{\Omega_1}{\sqrt{2}} - w_n = -\frac{\Omega_2}{\sqrt{2}}\frac{R_2}{R_1+R_2} - w_n$$

Spin at O_2

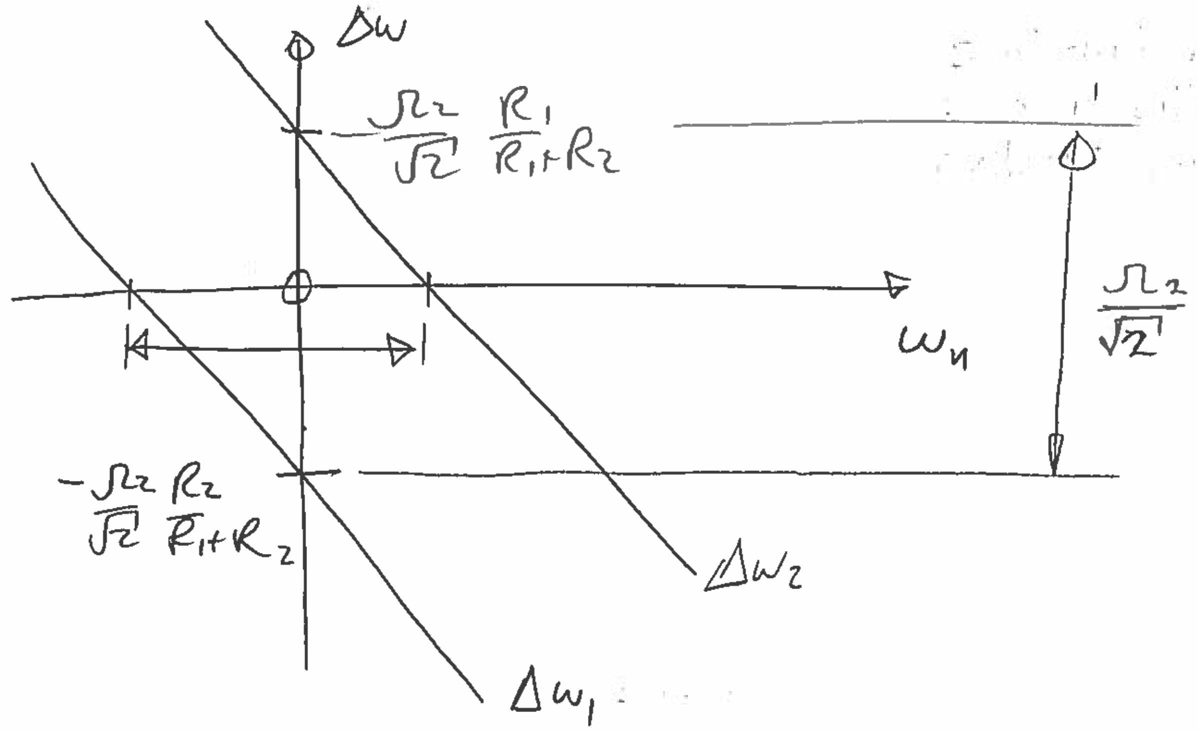
$$\Delta w_2 = \frac{\Omega_2 + \Omega_1}{\sqrt{2}} - w_n$$

$$= \frac{\Omega_2 - \Omega_2 \frac{R_2}{R_1+R_2}}{\sqrt{2}} - w_n$$

$$= \frac{\Omega_2}{\sqrt{2}}\left(1 - \frac{R_2}{R_1+R_2}\right) - w_n$$

$$= \frac{\Omega_2}{\sqrt{2}} \frac{R_1}{R_1+R_2} - w_n.$$

iii)

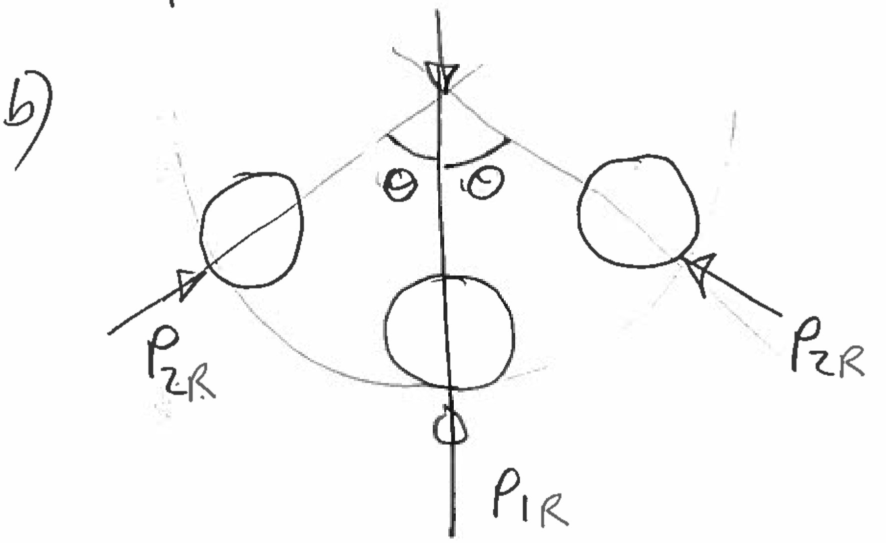


- Spin cannot be zero at both contacts simultaneously,
- likely value of ω_n is when [illegible]

$$-\frac{\Omega_2}{\sqrt{2}}\frac{R_2}{R_1+R_2} < \omega_n < \frac{\Omega_2}{\sqrt{2}}\frac{R_1}{R_1+R_2}$$

- Outside of this range, $|\Delta\omega_1| + |\Delta\omega_2|$ increases and implies some external power input to the ball.

b)



$\theta = 45^\circ$

Three balls in each bearing carry load

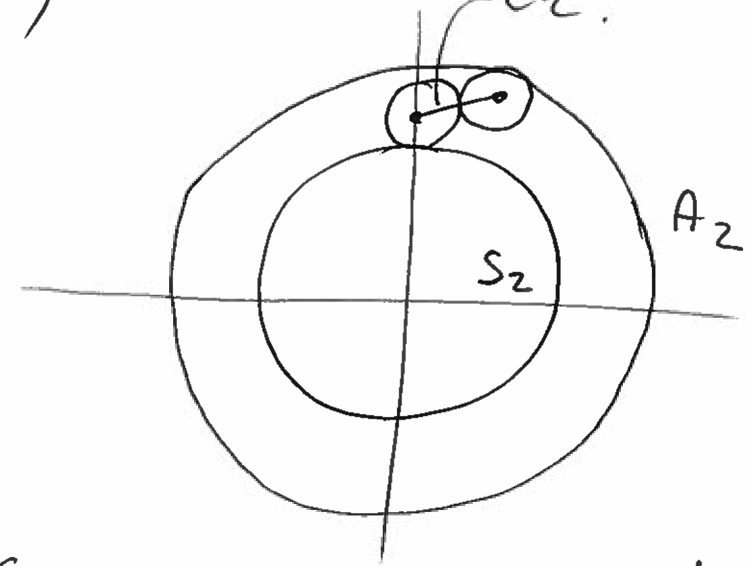
Assume that radial component of contact force varies as $\cos\theta$, so that total vertical force.

$$250N = P_{1R} + 2P_{1R}\cos^2 45^\circ$$

$$P_{1R} = 125N.$$

then normal contact forces are $\sqrt{2}$ times larger: $P_1 = 176{\cdot}8N$, $P_2 = 12{\cdot}5N$.

3. a)



Consider an arbitrary set of speeds $\omega_{S2}, \omega_{A2}, \omega_{C2}$

Superimpose $-\omega_{C2}$ to bring carrier to a halt

$$\frac{\omega_{S2}-\omega_{C2}}{\omega_{A2}-\omega_{C2}} = \frac{N_{A2}}{N_{S2}} = R_2$$

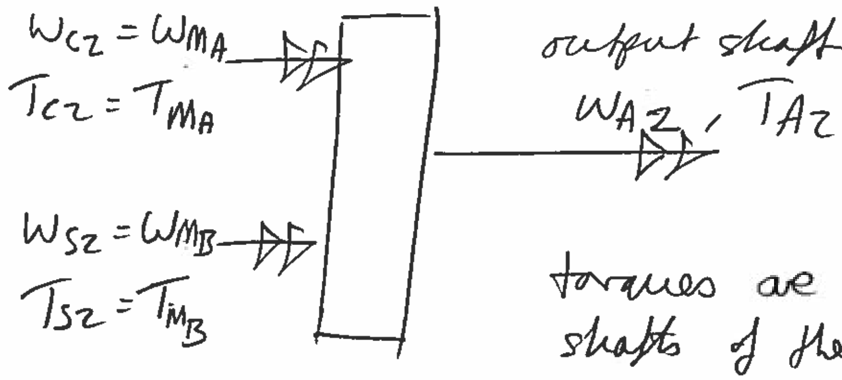
note +ve sign due to two planet wheels in series

$$\omega_{S2}-\omega_{C2} = (\omega_{A2}-\omega_{C2}) R_2$$

$$\underline{\underline{\omega_{S2} = \omega_{A2} R_2 + \omega_{C2}(1-R_2)}}$$

(same as data sheet but R has opposite sign)

b) Carrier 1 is free to rotate so epicyclic 1 contributes no torques, consider only epicyclic 2



torques are applied to the shafts of the epicyclic.

Use virtual power to find torque ratios

$$\omega_{C2}' T_{C2} + \omega_{S2}' T_{S2} + \omega_{A2}' T_{A2} = 0$$

want $\dfrac{T_{C2}}{T_{A2}}$ so let $\omega_{S2}' = 0$

$$\omega_{C2}' T_{C2} = -\omega_{A2}' T_{A2}$$

$$\frac{T_{C2}}{T_{A2}} = \frac{-\omega_{A2}'}{\omega_{C2}'} = \frac{1-R_2}{R_2} = \frac{1}{R_2} - 1$$

where $R_2 = \dfrac{N_{A2}}{N_{S2}} = \dfrac{90}{40}$ $\quad \therefore \dfrac{T_{C2}}{T_{A2}} = \dfrac{4}{9} - 1 = -\dfrac{5}{9}$

output torque $T_{A2} = 36\ Nm$ $\quad \therefore T_{C2} = -\dfrac{5}{9} \cdot 36\ Nm$

motor A torque $T_{MA} = T_{C2} = \underline{\underline{-20\ Nm}}$

torque equilibrium $T_{C2} + T_{S2} + T_{A2} = 0$

$$\therefore T_{MB} = T_{S2} = -T_{A2} - T_{C2}$$

motor B torque $T_{MB} = -36 + 20 = \underline{\underline{-16\ Nm}}$

speed of output shaft is same as speed of motors A and B, $\underline{600\ rad/s}$ (the whole epicyclic rotates as a rigid body)

c) carrier 1 is braked, output shaft speed = 60 rad/s

epicyclic speed rule for epicyclic 1 from data sheet

$$\omega_{S1} = (1+R_1)\omega_{C1} - R_1\omega_{A1} \qquad R_1 = \frac{N_{A1}}{N_{S1}} = \frac{90}{45} = 2$$

carrier 1 is braked, $\omega_{C1} = 0$

$$\therefore \quad \omega_{S1} = -R_1\omega_{A1} \quad \text{—— (1)}$$

linkages

$$\omega_{S1} = \omega_{S2} \quad \text{—— (2)}$$

$$\omega_{A1} = \omega_{C2} \quad \text{—— (3)}$$

speed rule for epicyclic 2 (part (a) of question)

$$\omega_{S2} = \omega_{A2}R_2 + \omega_{C2}(1-R_2) \qquad R_2 = \frac{9}{4}$$

using (1), (2), (3)

$$-R_1\omega_{A1} = \omega_{A2}R_2 + \omega_{A1}(1-R_2)$$

$$\omega_{A1}(-R_1 - 1 + R_2) = \omega_{A2}R_2$$

$$\omega_{A1} = \omega_{A2}\frac{R_2}{-R_1-1+R_2}$$

$\therefore$ motor A

$$\omega_{MA} = \omega_{A1} = \frac{60 \cdot 9/4}{-2-1+9/4}$$

$$\underline{\underline{\omega_{MA} = -180 \text{ rad/s}}}$$

motor B

$$\omega_{MB} = \omega_{S1} = -R_1\omega_{A1} \qquad \text{(epicyclic 1 speed rule)}$$

$$= -2 \cdot (-180)$$

$$\underline{\underline{\omega_{MB} = 360 \text{ rad/s}}}$$

To find torques, consider whether motors are operating at torque limit or power limit.

The transition speed is $\omega_{transition} \cdot 80 \text{ Nm} = 12 \text{ kW}$

$$\omega_{transition} = 150 \text{ rad/s}$$

Hence both motors are operating at power limit

Consider conservation of power through lossless gearbox to find output torque

$$12\,kW + 12\,kW = T_{A2} \cdot \omega_{A2}$$

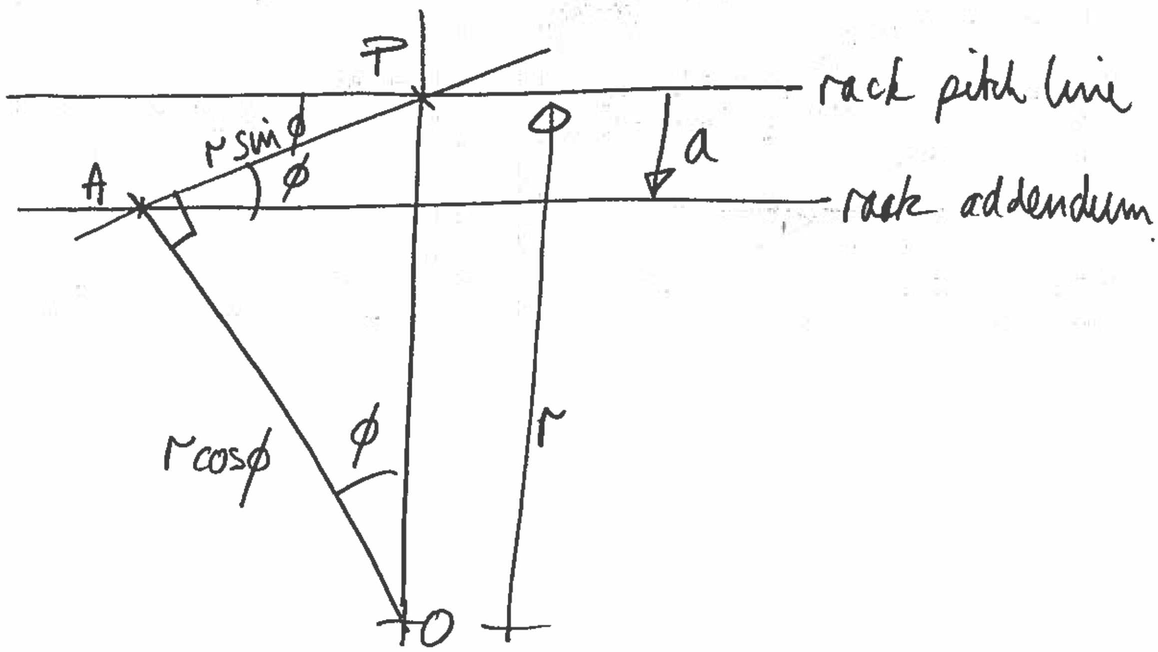
$$T_{A2} = \frac{24 \cdot 10^3\ W}{60\ rad/s}$$

$$\underline{\underline{T_{A2} = 400\ Nm}}$$

d) The arrangement allows the torque of the motors to be amplified at low output speeds. A fixed gear ratio would require a larger, heavier motor.

4. a)
- constant speed ratio
- gears of different diameters can be combined
- contact force acts along fixed axis (low noise)
- centre distance is not critical
- simple to manufacture

b) i)



The minimum number of teeth (or maximum module) occurs when the rack addendum intercepts point A, the tangent to the pinion base circle.

$$a = m = r\sin\phi \cdot \sin\phi$$

$$r = \frac{m}{\sin^2\phi}$$

where $r = \frac{mN}{2}$ $\quad N$ = number of pinion teeth

equate $\frac{mN}{2} = \frac{m}{\sin^2\phi}$

$$N = \frac{2}{\sin^2\phi} = 17.1 \qquad (\phi = 20^\circ)$$

Therefore $\underline{\underline{N = 18}}$

contact pressure

contact is a cylinder of radius $r\sin\phi$ on flat plate

data sheet: $p_0 = \sqrt{\dfrac{P'E^*}{\pi R}}$

where $P' = \dfrac{P_T}{w\cos\phi}$ $\quad P_T$ = rack force

and $R = r\sin\phi = \dfrac{Nm}{2}\sin\phi$

combine $p_0 = \sqrt{\dfrac{P_T\, E^*\, 2}{\pi\, w\cos\phi\, Nm\, \sin\phi}}$

$$p_0 = \sqrt{\frac{5.10^6 \,.\, 115.10^9 \,.\, 2}{\pi\ 0.127\ \cos 20 \,.\, 18 \,.\, 0.1 \,.\, \sin 20^\circ}}$$

$p_0 = \underline{\underline{2.23\ \text{GPa}}}$

torque

pinion torque $T = P_T r = P_T \dfrac{Nm}{2}$

$$= \frac{5.10^6 . 18 . 0.1}{2}$$

$T = \underline{\underline{4.5\ \text{MNm}}}$

ii)

$\phi = 25^\circ \quad \therefore N_{min} = \dfrac{2}{\sin^2\phi} = 11.2$

$N_{min} = \underline{\underline{12}}$

$p_0 = 2.23\ \text{GPa} \sqrt{\dfrac{18}{12} \cdot \dfrac{\cos 20\ \sin 20}{\cos 25\ \sin 25}}$

$p_0 = \underline{\underline{2.5\ \text{GPa}}}$

torque $T = P_T \frac{Nm}{2} = 5.10^6 . \frac{12 . 0.1}{2}$

$= 3 \text{ MNm}.$

iii) repeat (i) but with $a = \frac{2}{3} m$.

$\frac{2}{3} m = r \sin\phi \sin\phi$

$\therefore r = \frac{2}{3} \frac{m}{\sin^2\phi}$

equate to $\frac{\cancel{m} N}{2} = \frac{2}{3} \frac{\cancel{m}}{\sin^2\phi}$

$N = \frac{4}{3\sin^2\phi} = 11.4 \quad (\phi = 20^\circ)$

$N_{min} = 12$

only N changed, so $p_0 = 2.23 \text{GPa} \sqrt{\frac{18}{12}}$

$= 2.73 \text{ GPa}$

$T = P_T r = P_T \frac{Nm}{2} = 5.10^6 \frac{12 . 0.1}{2}$

$T = 3 \text{ MNm}$

c) Increasing pressure angle and rack addendum both allow smaller number of pinion teeth, hence reduced pinion diameter and reduced pinion torque.
Results of (ii) and (iii) give same reduction in torque, but the increased pressure angle gives a smaller increase in contact pressure than the increased rack addendum.
In practice there are constraints on contact pressure, bending stress, contact ratio (≥ 1) and rack addendum.