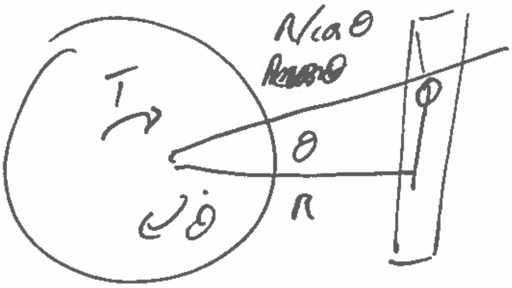
3C8 MACHINE DESIGN
ENGINEERING TRIPOS IIA 2022 (M SUTCLIFFE)

1. (a)

(This first element often missed)



Crossed cylinder geometry
⇒ circular contact patch with equivalent geometry of flat on sphere radius r (not r/2)

$$T = \frac{PR}{\cos\theta}$$ where P is contact force

$R \equiv r$
$E^* = \frac{1}{2}\frac{E}{1-\nu^2}$

From data sheet Hertz pressure $p_0 = \frac{1}{\pi}\left(\frac{6PE^{*2}}{R^2}\right)^{1/3}$

contact patch

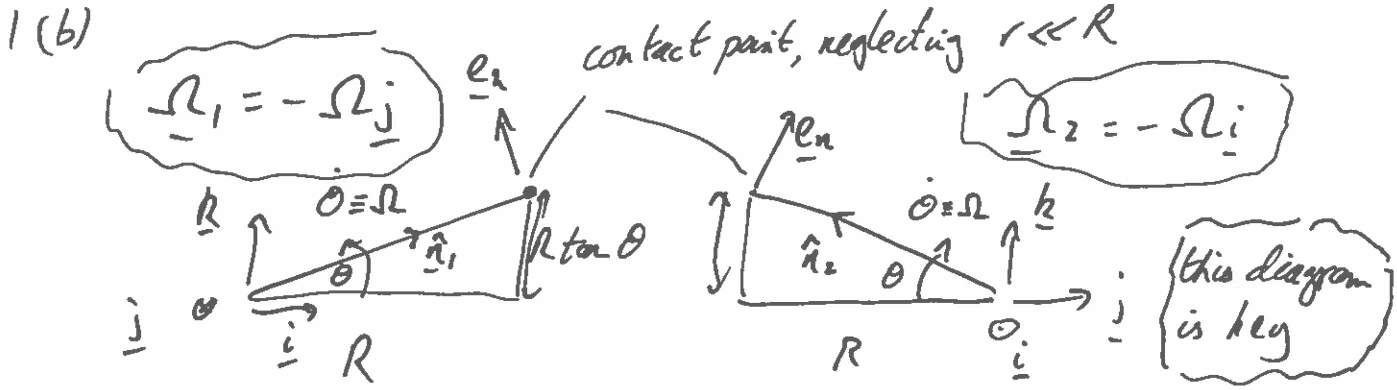
$$p_0 = \frac{1}{\pi}\left(\frac{3T\cos\theta}{r^2 R}\frac{E^2}{(1-\nu^2)^2}\right)^{1/3}$$

Contact pressure is semi-elliptical with peak stress p_0

Not as well done as expected.

(b) Examiner's comment. In hindset this would have been much better with a 'show that' part, or perhaps more help with the geometry. Many candidates struggled to get started on this and the next part, which required careful geometry. Different approximations about eg small θ were allowed.

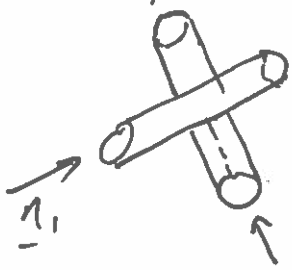
1 (b)



$\underline{e}_n$ is the common normal

View as Fig. 1 — View looking back along $\underline{i}$

As contact has same height $\underline{k}$ ($= R\tan\theta$), $\dot{\theta}$ is the same for both gears; call this Ω. (often missed)



The contact normal for the crossed cylinders setup is normal to the $\underline{n}$ vectors pointing along the axes of the two cylinders. (This contact mechanics point often missed)

Neglecting $r \ll R$ in the geometry:

$$\underline{e}_n = \hat{\underline{n}}_1 \times \hat{\underline{n}}_2 = \frac{R}{R\sqrt{1+\tan^2\theta}}\begin{pmatrix}1\\0\\\tan\theta\end{pmatrix} \times \frac{R}{R\sqrt{1+\tan^2\theta}}\begin{pmatrix}0\\-1\\\tan\theta\end{pmatrix}$$

$$= \frac{1}{(1+\tan^2\theta)}\begin{pmatrix}\tan\theta\\-\tan\theta\\-1\end{pmatrix}$$

Unit vector $\hat{\underline{e}}_n = \begin{pmatrix}\tan\theta\\-\tan\theta\\-1\end{pmatrix}\frac{1}{\sqrt{1+2\tan^2\theta}} = \begin{pmatrix}\sin\theta\\-\sin\theta\\-\cos\theta\end{pmatrix}\frac{1}{\sqrt{1+\sin^2\theta}}$

Angle ϕ between $\hat{\underline{e}}_n$ and $\underline{k}$: $\cos\phi = \hat{\underline{e}}_n \cdot \underline{k}$

$$\phi = \cos^{-1}\left(\frac{1}{\sqrt{1+2\tan^2\theta}}\right) \quad (\pm \text{ angle})$$

1 (c) (i)

The two pegs have the same component along the common normal $\hat{\underline{e}}_n$, the difference gives the sliding speed:

$$\underline{\Delta v} = \underline{v}_1 - \underline{v}_2 \qquad (\text{or } \underline{v}_2 - \underline{v}_1)$$

Here $\underline{v}_1 = \underline{\omega} \times \underline{r} = -\Omega \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times R \begin{pmatrix} 1 \\ 0 \\ \tan\theta \end{pmatrix} = \Omega R \begin{pmatrix} -\tan\theta \\ 0 \\ 1 \end{pmatrix}$

$$\underline{v}_2 = -\Omega R \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ \tan\theta \end{pmatrix} = \Omega R \begin{pmatrix} 0 \\ \tan\theta \\ 1 \end{pmatrix}$$

$$\underline{v}_1 - \underline{v}_2 = \Omega R \begin{pmatrix} -\tan\theta \\ -\tan\theta \\ 0 \end{pmatrix}$$

Check: $\underline{v}_1 \cdot \hat{\underline{e}}_n = (-\tan^2\theta - 1) \dfrac{\Omega R}{\sqrt{1+2\tan^2\theta}}$

$\underline{v}_2 \cdot \hat{\underline{e}}_n = (-\tan^2\theta - 1) \dfrac{\Omega R}{\sqrt{1+2\tan^2\theta}}$

✓ Equal as required

(ii) Spin velocity $\underline{\omega}_2 = \omega_2 \hat{\underline{e}}_n$

$$\omega_2 = (\underline{\Omega}_1 - \underline{\Omega}_2) \cdot \hat{\underline{e}}_n \qquad (\text{or } \Omega_2 - \Omega_1)$$

$$= \left[-\Omega \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \Omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} \tan\theta \\ -\tan\theta \\ -1 \end{pmatrix} \frac{1}{\sqrt{1+2\tan^2\theta}}$$

$$\omega_2 = \frac{-2\tan\theta \cdot \Omega}{\sqrt{1+2\tan^2\theta}}$$

$$\underline{\omega}_2 = \omega_2 \hat{\underline{e}}_n$$

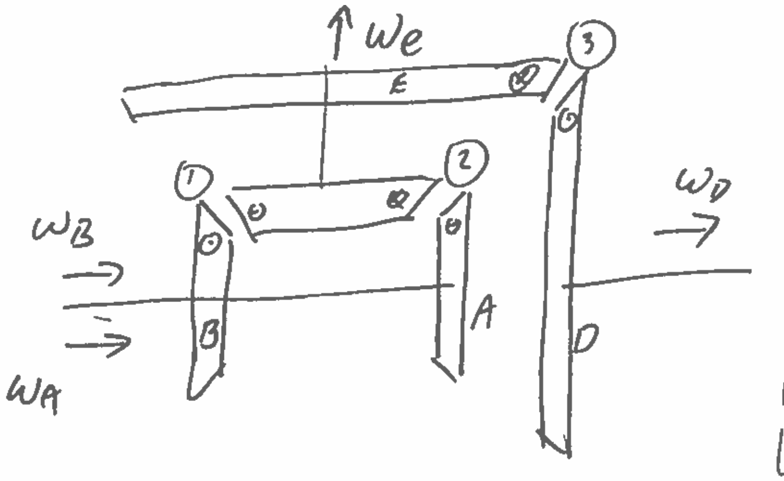
These two parts were difficult to do without the geometry of (b) in place

1(d) Power loss depends on contact force, friction force and sliding speed.

The contact force is shown in (a) to vary with $\cos\theta$, so falling with increasing θ.

But the sliding speed and spin velocity both vary as $\tan\theta$, so (assuming small θ) the net effect will be an increase in power lost due to friction with increasing θ.

2.(a)



Use ⊙ ⊗ notation to get signs right. Define direction of $\underline{\omega_e}$

① $\omega_b \cdot b = \omega_e e_2$ (Tooth numbers proportional to radii)

② $\omega_A \cdot a = -\omega_e \cdot e_2$

③ $\omega_D \cdot d = -\omega_e \cdot e_1$

$$\Rightarrow \frac{\omega_A}{\omega_A} = -\frac{a}{b} = -1$$

The sign was often incorrect

$$\frac{\omega_D}{\omega_A} = \frac{a}{b}\frac{e_1}{e_2} = \lambda$$

$\lambda \equiv \frac{a e_1}{b e_2}$ helpful to ease working

(b) Use tabular approach

Don't need ω_E

ω_A	ω_B	ω_C	ω_D	
X	$-X$	0	λX	⇐ this from A
$X+Y$	$-X+Y$	Y	$\lambda X+Y$	⇐ superimpose Y on everything

Avoid using ω_A inside table which caused confusion

$\Rightarrow 2Y = \omega_A + \omega_B$; $2X = \omega_A - \omega_B$

$$\omega_D = \lambda X + Y = \frac{\lambda}{2}(\omega_A - \omega_B) + \frac{1}{2}(\omega_A + \omega_B) = \frac{\lambda+1}{2}\omega_A + \frac{\lambda-1}{2}\omega_B$$

In form as required with $\alpha = (\lambda+1)/2$, $\beta = (\lambda-1)/2$

(c) $\frac{P_A}{P_D} = \frac{\omega_A T_A}{\omega_D T_D}$. Now $\omega_B = \omega_A/2 \Rightarrow$ ~~$\omega_D = \ldots \omega_A + \ldots \omega_A$~~

$$\Rightarrow \omega_D = \alpha\omega_A + \beta\frac{\omega_A}{2} \qquad \frac{\omega_D}{\omega_A} = \alpha + \beta/2$$

Much easier using virtual power

Using virtual work to get T_A/T_D

$$T_A\omega_A' + T_B\omega_B' + T_D\omega_D' = 0 \ : \text{put } \omega_B' = 0 \Rightarrow \frac{T_A}{T_D} = -\frac{\omega_D'}{\omega_A'}\Bigg|_{\omega_B'=0} = -\alpha$$

Combining $\frac{P_A}{P_D} = \frac{\omega_A}{\omega_D}\frac{T_A}{T_D} = \frac{-\alpha}{\alpha + \beta/2}$

$$\frac{\text{Power in}}{\text{Power out}} = \frac{+\alpha}{\alpha+\beta/2}$$

Relatively straightforward question. Simplify algebra by introducing λ and working in terms of α and β

3(a) (i) $F = mgC_R + \frac{1}{2}\rho A C_D V^2$

$= 1500 \cdot 9{\cdot}81 \cdot 0{\cdot}02 + \frac{1}{2} \cdot 1{\cdot}2 \cdot 2{\cdot}0 \cdot 0{\cdot}3 \cdot V^2$

$F = 296{\cdot}3 + 0{\cdot}36 V^2$

(need to spot)

Max power of motor at corner of operating curve (4500, 125)

$\text{Power} = 125\,\text{Nm} \times \frac{4500}{60} \times 2\pi \text{ rad/s} = 58{\cdot}9\,\text{kW}$

(471.2 rad/s)

Max vehicle speed at maximum power

$\Rightarrow \text{Power} = 58{\cdot}9 \times 10^3 = F \cdot V_{max} = 296{\cdot}3\, V_{max} + 0{\cdot}36\, V_{max}^3$

Find by iteration/calculator $V_{max} \simeq 50$ m/s

wheel speed Ω, motor speed ω

$\Rightarrow V_{max} = \Omega R \Rightarrow \Omega = \frac{50 \text{ m/s}}{0{\cdot}35 \text{ m}} = 142{\cdot}9$ rad/s

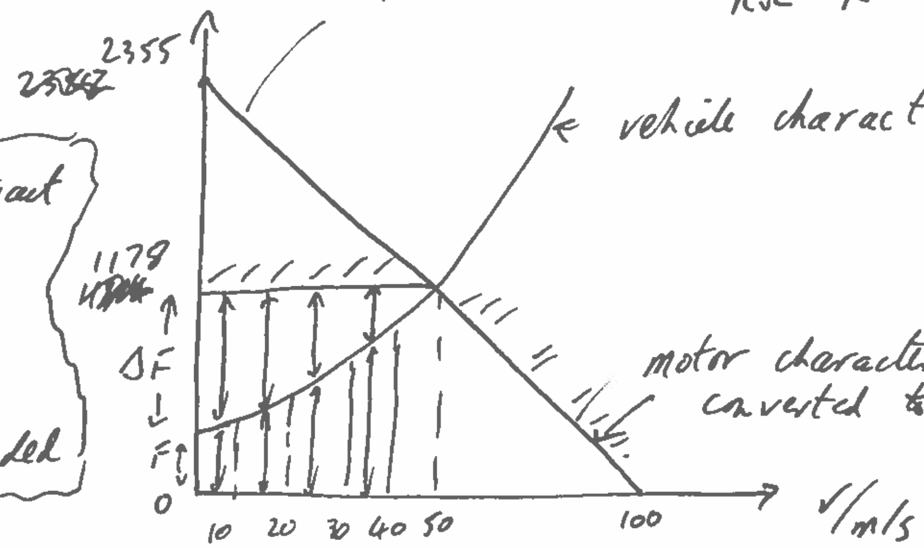
$\frac{\omega}{\Omega} = \frac{\text{motor speed}}{\text{wheel speed}} = G_1 = \frac{471{\cdot}2}{142{\cdot}9} = 3{\cdot}297 \simeq 3{\cdot}30$

{This part reasonably well done}

3 (a) (ii)

For motor

$FV = T\omega \Rightarrow F = \frac{T\omega}{R\Omega} = \frac{T}{R}G_1 = T \times \frac{3.297}{0.35}$



Need to extract ΔF, not F – often missed, especially if no sketch included

Need to find extra torque ΔT and hence extra force ΔF available for acceleration $\Delta F = 1178 - F$

⇒ put vehicle and motor characteristics on same graph

This part critical and often missed

Numerical calculation up to 80% of 50 m/s → 40 m/s

$\Delta F = ma = m\frac{\Delta v}{\Delta t} \Rightarrow \Delta t = \frac{m\,\Delta v}{\Delta F}$ ‖ split into 10s intervals

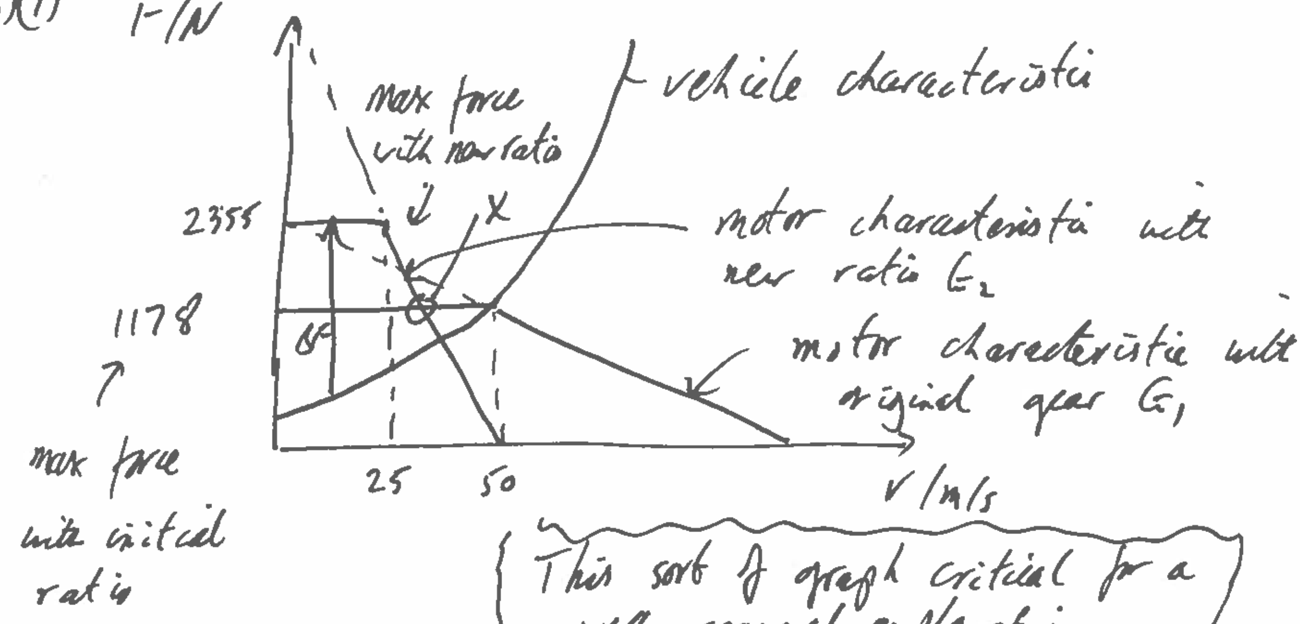
value at middle of 0→10 m/s interval

v / m/s	5	15	25	35
F / N	303	375	519	735
(1178 − F) → ΔF / N	875	803	659	443
Δv / m/s	10	10	10	10
$\left(\frac{m\,\Delta v}{\Delta F}\right)$ Δt / s	17.1	18.7	22.8	33.9

$\Sigma \Delta t = 92.5\,s$. **Answer ≃ 90 s**

Any reasonable iteration scheme OK. Many candidates fell down by not writing down the few critical equations on this page.

3(b)(i)



{This sort of graph critical for a well-reasoned explanation}

By introducing a new gear the available force ΔF between the motor characteristic and the road vehicle characteristic can be increased at lower speeds, at the expense of reduced ΔF at higher speeds (and a lower maximum speed in this new ratio G_2).

As a compromise / first estimate, choose the maximum power point for G_2 at half that for G_1, i.e. at 25 m/s. ~~Then the~~

This corresponds to a doubling of G_1, $\underline{G_2 = 6.60}$

The cross-over point when G_2 delivers less ΔF than G_1 is labelled as X on the graph.

Maybe a slightly higher value of G would be better, really only a numerical study will give an optimum.

3 (b) (ii)

Cross-over is at point X with speed $25 + \frac{25}{2} \approx 37.5$ m/s (from graph)

Need to update Δt calculation

	0–10	10–20	20–30	30–40
v / m/s	5	15	25	35
original ΔF	875	803	659	443
extra ΔF / N	1177	1177	~1000	~400
new Δt / s	7.3	7.6	9.0	17.8

~400: estimate to allow for uplift between 30 and 37.5 m/s

~1000: reduced a bit to allow for drop off after 25 m/s

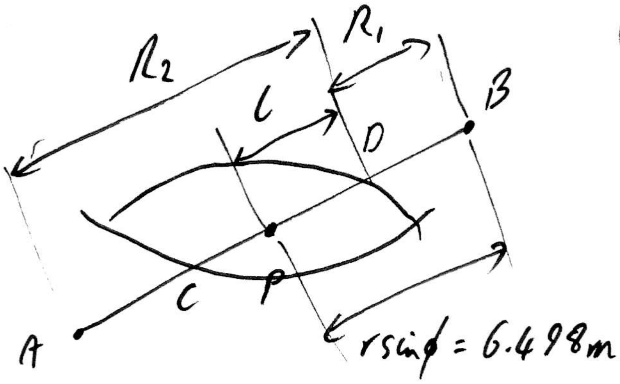
$$\Delta t = \frac{m \Delta v}{\text{original } \Delta F + \text{extra } \Delta F}$$

New $\Sigma \Delta t = 41.7$ s ≈ 40 s

People without a sketch tended to struggle with this final part

Q 4(a)

SPUR GEARS



Critical to spot this consequence of $r_c = 2$

Since $r_c = 2$, $l = p_b = \pi m \cos\phi = 2.952m$

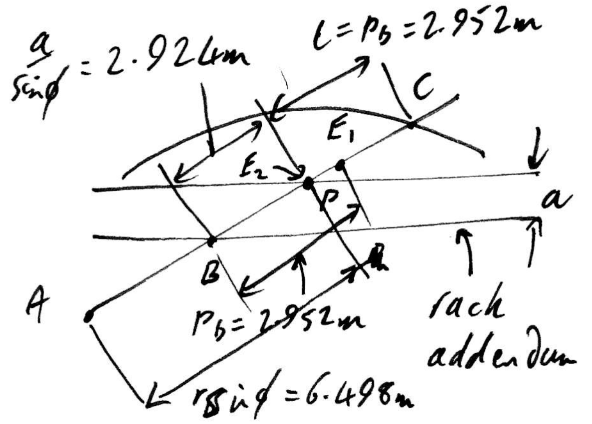
$r = \frac{mN}{2} = 19m$

Critical case of 2 contacts at C and D (since $r_c = 2$)

$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{3.546m} + \frac{1}{9.450m}$

$\Rightarrow R = 2.579m$

RACK



Limits of single pair contact between E_1 and E_2. Noting that $l = p_b$, E_2 coincides with P. This location gives the smallest radius of contact for single contact

$R_{E2} = r\sin\phi = 6.498m$

Double contact:

$R_B = 6.498m - 2.924m = 3.574m$

Single contact worse since $2R_B > R_{E1}$

Same surface failure mechanism

$$\Rightarrow \frac{(p_0)_{SS}}{(p_0)_{RS}} = 1 = \frac{\left(\frac{P'}{2R}\right)^{\frac{1}{2}}_{SS}}{\left(P'/R\right)^{\frac{1}{2}}_{RS}}$$

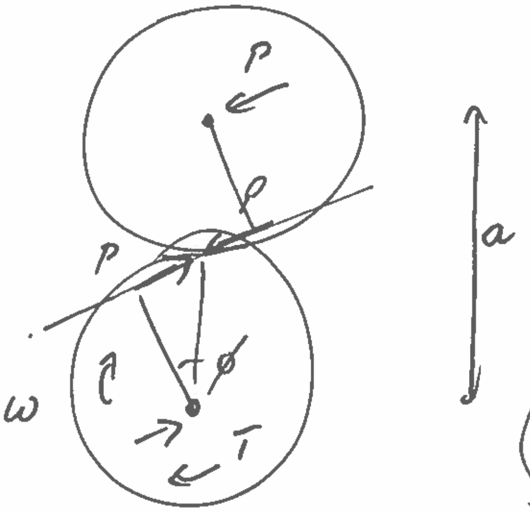
(SS = SPUR, RS = RACK; $\frac{P'}{2}$: 2 contacts)

Since the same pinion with the same wear is used $P' \propto T$

i.e. $\frac{T_{RS}}{T_{SS}} = \frac{R_{RS}}{2R_{SS}} = \frac{6.498}{2 \times 2.579} = 1.26$

People who draw the two figures and spotted the implication of $r_c = 2$ for the spur/spur contact answered this well.

Q 4 (b)(i)



Bearing force = P

Power $Q = T\omega = \frac{Pa}{2}\cos\phi\,\omega$

> This sketch relating power to gear geometry and P was often missed

$$\text{Bearing life } L_{10} = \left(\frac{C}{P}\right)^3 \quad \text{since } a_1 = a_{23} = 1$$

We want to minimise a, subject to $a \geq 1.1D$

$$C \geq L_{10}^{\frac{1}{3}} P = L^{\frac{1}{3}} \frac{2Q}{a\cos\phi\,\omega} \Rightarrow Ca \geq \frac{L^{\frac{1}{3}}\, 2Q}{\cos\phi\,\omega} \Rightarrow CD \geq \frac{L_{10}^{\frac{1}{3}}\, 2Q}{1.1\cos\phi\,\omega}$$

So lifetime calc sets a constraint on CD, and to minimise a put $a = 1.1D$ and minimise D.

> This logic not always set out clearly by students

(ii)

$$CD \geq \frac{\left(\frac{2000.60.100}{10^6}\right)^{\frac{1}{3}} . 2 . 50\times10^3}{1.1 . \cos 20° . \left(\frac{2000\times2\pi}{60}\right)} = 1058 \text{ Nm}$$

Need to choose bearing from datasheet with smallest D but $CD \geq 1058$

increasing D ↓

D /mm	C /N	CD /Nm	bearing
47	4750	223	61807
52	4940	257	61808
55	9560	526	61907
58	6050	351	61809
62	15,900	986	6007
65	6,240	406	61810
68	16,800	1142	6008 ←

for given D choose bearing with largest D

> Some sensible use of datasheet attracted marks

Hence select bearing 6008