EGT2
ENGINEERING TRIPOS PART IIA

Monday 9 May 2022 2.00 to 3.40

**Module 3C9**

**FRACTURE MECHANICS OF MATERIALS AND STRUCTURES**

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

**STATIONERY REQUIREMENTS**
Single-sided script paper

**SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM**
CUED approved calculator allowed
Attachment: 3C9 Fracture Mechanics of Materials and Structures (8 pages)
Engineering Data Book

**10 minutes reading time is allowed for this paper at the start of the exam.**

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.**

**You may not remove any stationery from the Examination Room.**

1 (a) Distinguish between the energy release rate $\mathcal{G}$ and the stress intensity factor $K$ as loading parameters at a crack tip. [15%]

(b) Distinguish between the stress states directly ahead of an isolated sharp crack of length $2a$, and an isolated notch of length $2a$ and height $2b$, in an infinite thin sheet under a remote tensile stress $\sigma^{\infty}$. Both the crack and notch are transverse to the direction of loading, and the stress concentration factor for the notch is given by $K_t = 1 + (2a/b)$. [15%]

(c) Two steel blocks, each of width $2W$ and height $2H$, are adhesively bonded by an epoxy layer of height $h << H$, as shown in Fig 1. The unit thickness of the adhesive (into the page) is much less than the height $h$. Assume that the steel is rigid and the epoxy is a linear elastic, isotropic solid of Young's modulus $E$ and Poisson ratio $\nu$. The epoxy layer contains a centre crack of length $2a$, and the upper block is displaced by a shear displacement $u$ and by a normal displacement $w$ with respect to the lower block, giving rise to a shear force $S$ and normal force $N$.

(i) Determine the stored elastic energy in the epoxy layer assuming that the epoxy is in a uniform state of tension and shear. Explain the approximations that are inherent in this assumption. [30%]

(ii) Obtain an expression for the energy release rate $\mathcal{G}$ at each crack tip, for crack advance at fixed load. Sketch the anticipated crack path. [40%]

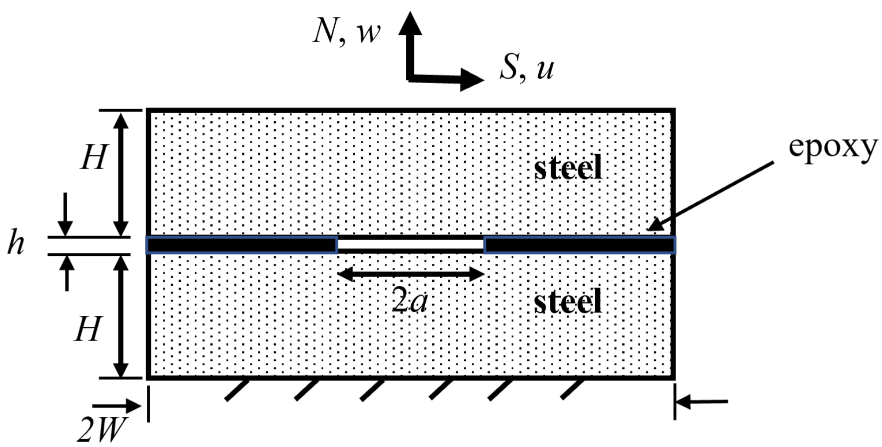


Fig. 1

2 (a) Explain why the mode I and mode II toughnesses are comparable in magnitude for cleavage of a ceramic, whereas the mode I and mode II toughnesses may be very different in magnitude for ductile fracture of a metallic alloy. [40%]

(b) A double cantilever beam, of depth $B$ into the page, is of geometry shown in Fig. 2, and is used for performing a toughness test on ceramics and on steels.

(i) Assume that linear elastic fracture mechanics prevails for a ceramic specimen. Calculate the energy release rate $\mathcal{G}$ and hence the mode I stress intensity factor $K$ for the cantilever beam under a fixed end load $P$. [30%]

(ii) Consider instead a tough steel. Calculate the collapse limit load $P_L$ as a function of crack length assuming that the steel behaves in a rigid, ideally plastic manner with a yield strength $\sigma_y$. For a fixed end displacement $u = u_0$ applied to each arm of the beam, obtain an expression for the value of the J-integral. [30%]

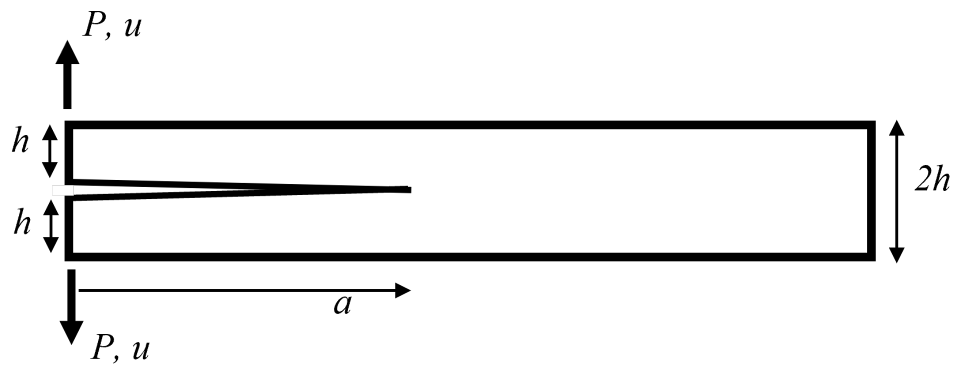


Fig. 2

3 A large welded steel plate contains an in-plane, residual tensile stress equal to the yield strength $\sigma_y$ over a width $2w$, as shown in Fig. 3. In addition, the plate is subjected to a cyclic in-plane compressive stress $\sigma$ which varies from zero to $-\sigma_y/2$. A through-crack of length $2a$ exists in the middle of the weld-run, as shown.

(a) Obtain expressions for the maximum stress intensity factor $K_{max}$, minimum stress intensity factor $K_{min}$, and the stress intensity range $\Delta K$, as a function of crack semi-length $a$ for $a < w$ and for $a > w$. [30%]

(b) Sketch the dependence of $K_{min}$ and $K_{max}$ upon $a$, with salient points marked on the plot, including their values at $a = w$, and the value of $a/w$ for which $K_{min} = 0$. [30%]

(c) Suppose that the crack grows by fatigue at a rate $da/dN$ given by

$$\frac{da}{dN} = C(\Delta K)^n$$

where $C$ and $n$ are material constants. Obtain an expression for the number of cycles to grow the crack from a semi-length $a = w/10$ to $a = w$.

[30%]

(d) Explain the effect of a post-weld stress relief upon the fatigue life. [10%]

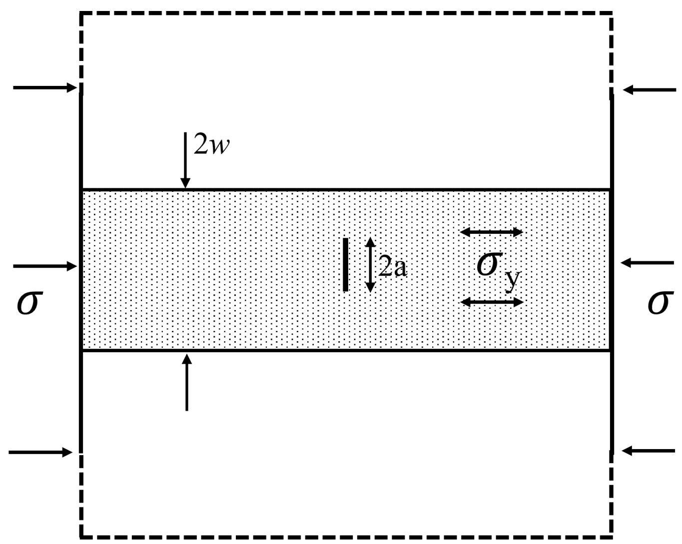


Fig. 3

4 Give a physical explanation for the following observations.

(a) The tensile strength and ductility of a scratched glass rod are much less than those of a scratched bar made from an aluminium alloy, despite the fact that they have comparable values of Young's modulus. [25%]

(b) A long fibre composite made from carbon fibres in an epoxy matrix has a toughness which much exceeds the rule-of-mixtures estimate. [25%]

(c) The toughness of a metallic aircraft wingskin, made by adhesive bonding together multiple thin sheets, exceeds that of a single sheet of thickness equal to that of the stack. [25%]

(d) The toughness of a pressure vessel steel can be adequately measured using small specimens, despite the fact that small scale yielding is violated for the small specimens. [25%]

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## ENGINEERING TRIPOS PART IIA

## Module 3C9 – FRACTURE MECHANICS OF MATERIALS AND STRUCTURES

## DATASHEET

### Crack tip plastic zone sizes

$$\text{diameter, } d_p = \begin{cases} \dfrac{1}{\pi}\left(\dfrac{K_I}{\sigma_y}\right)^2 & \text{Plane stress} \\[2ex] \dfrac{1}{3\pi}\left(\dfrac{K_I}{\sigma_y}\right)^2 & \text{Plane strain} \end{cases}$$

### Crack opening displacement

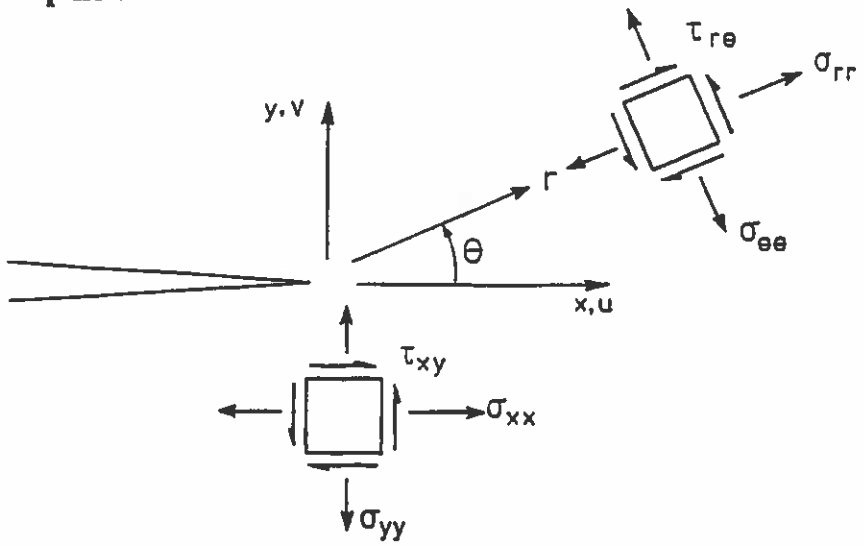
$$\delta = \begin{cases} \dfrac{K_I^2}{\sigma_y E} & \text{Plane stress} \\[2ex] \dfrac{1}{2}\dfrac{K_I^2}{\sigma_y E} & \text{Plane strain} \end{cases}$$

### Energy release rate

$$G = \begin{cases} \dfrac{1}{E}K_I^2 & \text{Plane stress} \\[2ex] \dfrac{1-\nu^2}{E}K_I^2 & \text{Plane strain} \end{cases}$$

Related to compliance $C$: $G = \dfrac{1}{2}\dfrac{P^2}{B}\dfrac{dC}{da}$

**Asymptotic crack tip fields in a linear elastic solid**




Mode I

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\ \sin\frac{\theta}{2}\ \cos\frac{3\theta}{2}$$

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{3\theta}{2}\right)$$

$$u = \begin{cases} \dfrac{K_I}{G}\sqrt{\dfrac{r}{2\pi}}\left(\dfrac{1-\nu}{1+\nu}+\sin^2\dfrac{\theta}{2}\right)\cos\dfrac{\theta}{2} & \text{Plane stress} \\ \dfrac{K_I}{G}\sqrt{\dfrac{r}{2\pi}}\left(1-2\nu+\sin^2\dfrac{\theta}{2}\right)\cos\dfrac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$v = \begin{cases} \dfrac{K_I}{G}\sqrt{\dfrac{r}{2\pi}}\left(\dfrac{2}{1+\nu}-\cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2} & \text{Plane stress} \\ \dfrac{K_I}{G}\sqrt{\dfrac{r}{2\pi}}\left(2-2\nu-\cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$w = 0$$

**Crack tip stress fields (cont'd)**

Mode II

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\frac{3\theta}{2}$$

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = -\frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

$$u = \begin{cases} \dfrac{K_{II}}{G}\sqrt{\dfrac{r}{2\pi}}\left(\dfrac{2}{1+\nu} + \cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2} & \text{Plane stress} \\[2ex] \dfrac{K_{II}}{G}\sqrt{\dfrac{r}{2\pi}}\left(2 - 2\nu + \cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$v = \begin{cases} \dfrac{K_{II}}{G}\sqrt{\dfrac{r}{2\pi}}\left(\dfrac{\nu-1}{1+\nu} + \sin^2\dfrac{\theta}{2}\right)\cos\dfrac{\theta}{2} & \text{Plane stress} \\[2ex] \dfrac{K_{II}}{G}\sqrt{\dfrac{r}{2\pi}}\left(-1 + 2\nu + \sin^2\dfrac{\theta}{2}\right)\cos\dfrac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$w = 0$$

Mode III

$$\tau_{zx} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}$$

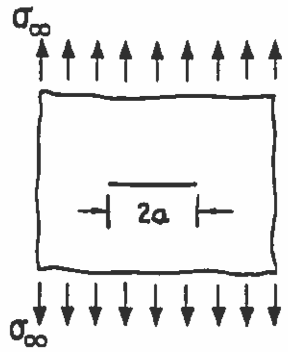
$$\tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

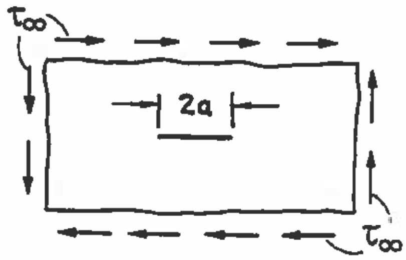
$$w = \frac{K_{III}}{G}\sqrt{\frac{2r}{\pi}} \sin\frac{\theta}{2}$$

$$u = v = 0$$

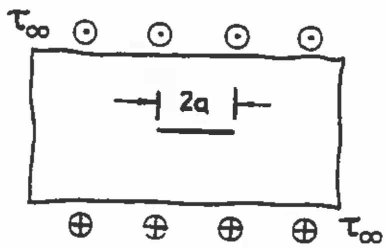
## Tables of stress intensity factors



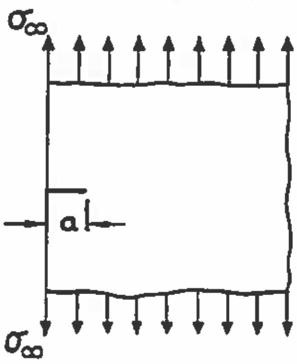

$$K_I = \sigma_\infty \sqrt{\pi a}$$



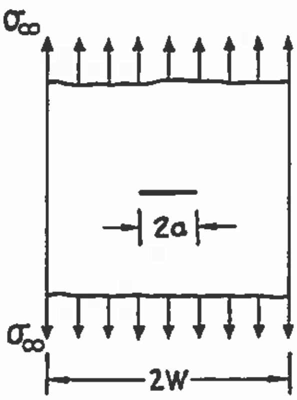

$$K_{II} = \tau_\infty \sqrt{\pi a}$$



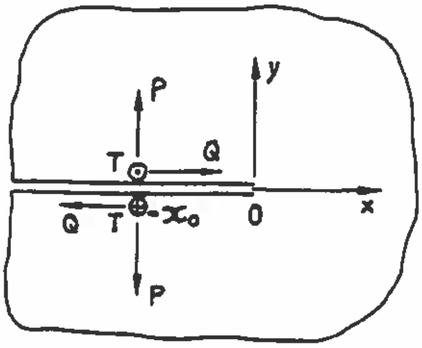

$$K_{III} = \tau_\infty \sqrt{\pi a}$$




$$K_I = 1.12\, \sigma_\infty \sqrt{\pi a}$$



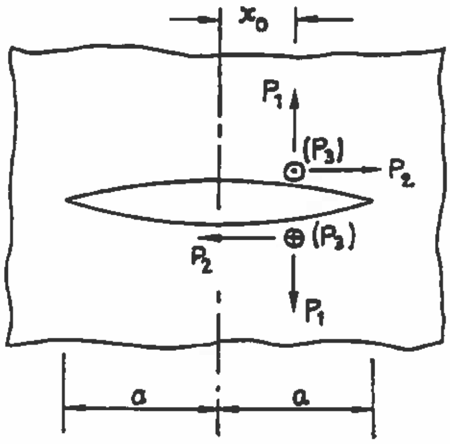

$$K_I = \sigma_\infty \sqrt{\pi a} \left( \frac{1 - a/2W + 0.326a^2/W^2}{\sqrt{1 - a/W}} \right)$$




$$K_I = \frac{2P}{\sqrt{2\pi x_o}}$$

$$K_{II} = \frac{2Q}{\sqrt{2\pi x_o}}$$

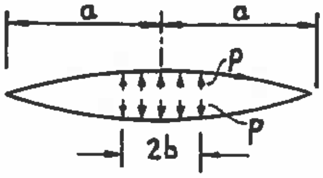
$$K_{III} = \frac{2T}{\sqrt{2\pi x_o}}$$



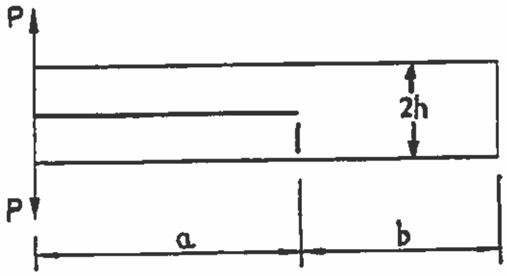

$$K_I = \frac{P_1}{\sqrt{\pi a}}\sqrt{\frac{a+x_o}{a-x_o}}$$

$$K_{II} = \frac{P_2}{\sqrt{\pi a}}\sqrt{\frac{a+x_o}{a-x_o}}$$

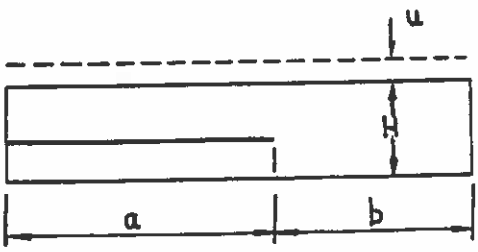
$$K_{III} = \frac{P_3}{\sqrt{\pi a}}\sqrt{\frac{a+x_o}{a-x_o}}$$




$$K_I = \frac{2pb}{\sqrt{\pi a}}\frac{a}{b}\arcsin\frac{b}{a}$$

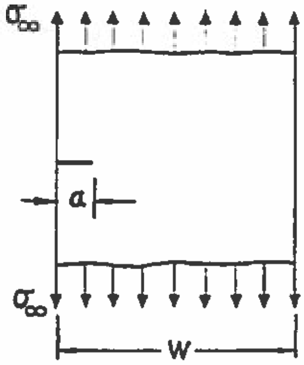



$$K_I = \frac{2\sqrt{3}}{h\sqrt{h}}\frac{Pa}{B} \qquad h \ll a \text{ and } h \ll b$$



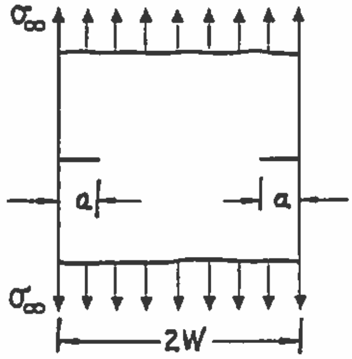

$$K_I = \sqrt{\frac{1}{2\alpha H}}\,Eu \qquad H \ll a \text{ and } H \ll b$$

$$\alpha = \begin{cases} 1-\nu^2 & \text{Plane stress} \\ 1-3\nu^2-2\nu^3 & \text{Plane strain} \end{cases}$$

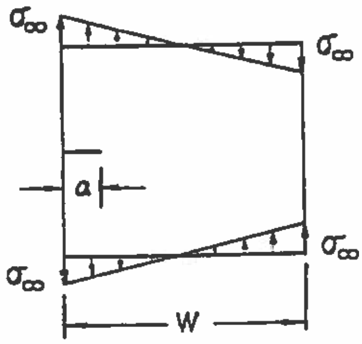



$a/W < 0.7$

$$K_I = \sigma_\infty \sqrt{\pi a}\left(1.12 - 0.23\frac{a}{W} + 10.6\frac{a^2}{W^2} - 21.7\frac{a^3}{W^3} + 30.4\frac{a^4}{W^4}\right)$$

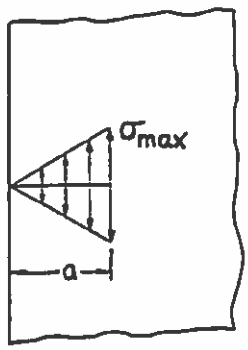



$$K_I = \sigma_\infty \sqrt{\pi a}\left(\frac{1.12 - 0.61a/W + 0.13a^3/W^3}{\sqrt{1 - a/W}}\right)$$

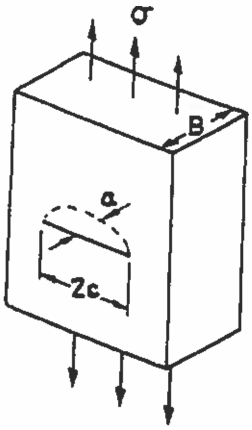



$a/W < 0.7$

$$K_I = \sigma_\infty \sqrt{\pi a}\left(1.12 - 1.39\frac{a}{W} + 7.3\frac{a^2}{W^2} - 13\frac{a^3}{W^3} + 14\frac{a^4}{W^4}\right)$$

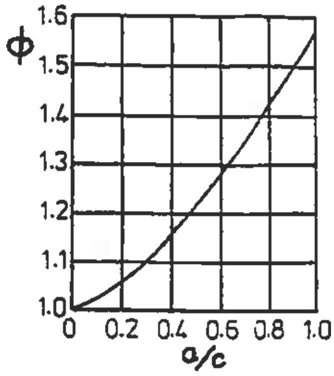


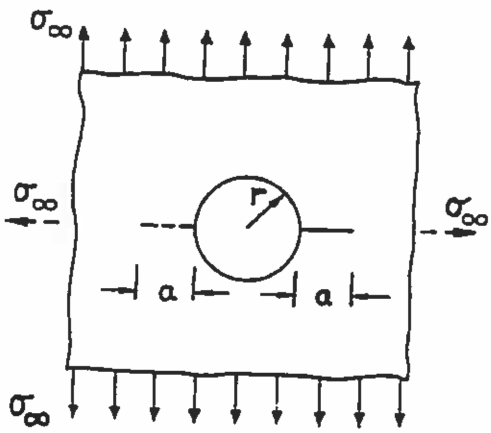

$$K_I = 0.683\ \sigma_{max}\sqrt{\pi a}$$




$$K_I = \frac{1.12}{\Phi}\sigma\sqrt{\pi a}$$

$$\Phi = \int_0^{\pi/2}\left(1-\frac{c^2-a^2}{c^2}\sin^2\theta\right)^{\frac{1}{2}}d\theta$$



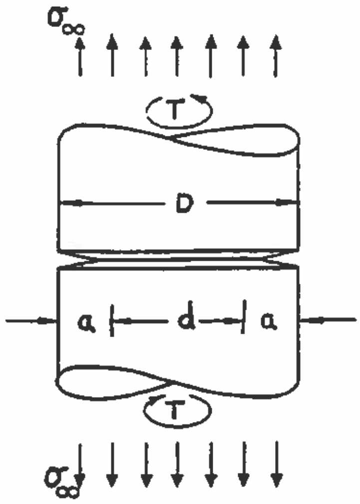




$$K_I = \sigma_\infty\sqrt{\pi a}\ F\left(\frac{a}{r}\right)$$

value of $F(a/r)$†

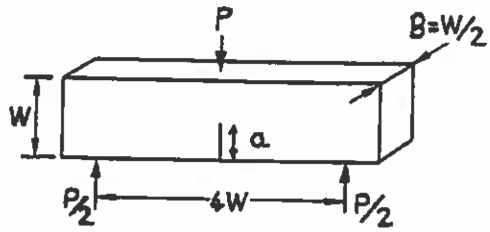
| $\frac{a}{r}$ | One crack U | One crack B | Two cracks U | Two cracks B |
|---|---|---|---|---|
| 0.00 | 3.36 | 2.24 | 3.36 | 2.24 |
| 0.10 | 2.73 | 1.98 | 2.73 | 1.98 |
| 0.20 | 2.30 | 1.82 | 2.41 | 1.83 |
| 0.30 | 2.04 | 1.67 | 2.15 | 1.70 |
| 0.40 | 1.86 | 1.58 | 1.96 | 1.61 |
| 0.50 | 1.73 | 1.49 | 1.83 | 1.57 |
| 0.60 | 1.64 | 1.42 | 1.71 | 1.52 |
| 0.80 | 1.47 | 1.32 | 1.58 | 1.43 |
| 1.0 | 1.37 | 1.22 | 1.45 | 1.38 |
| 1.5 | 1.18 | 1.06 | 1.29 | 1.26 |
| 2.0 | 1.06 | 1.01 | 1.21 | 1.20 |
| 3.0 | 0.94 | 0.93 | 1.14 | 1.13 |
| 5.0 | 0.81 | 0.81 | 1.07 | 1.06 |
| 10.0 | 0.75 | 0.75 | 1.03 | 1.03 |
| ∞ | 0.707 | 0.707 | 1.00 | 1.00 |

†$U$ = uniaxial $\sigma_\infty$  $B$ = biaxial $\sigma_\infty$.

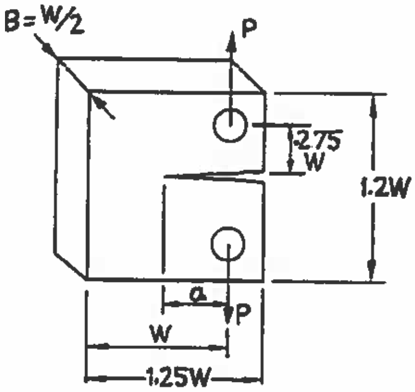



$$K_I = \sigma_\infty \sqrt{\pi a}\left(\frac{D}{d} + \frac{1}{2} + \frac{3}{8}\frac{d}{D} - 0.36\frac{d^2}{D^2} + 0.73\frac{d^3}{D^3}\right)\frac{1}{2}\sqrt{\frac{D}{d}}$$

$$K_{III} = \frac{16T}{\pi D^3}\sqrt{\pi a}\left(\frac{D^2}{d^2} + \frac{1}{2}\frac{D}{d} + \frac{3}{8} + \frac{5}{16}\frac{d}{D} + \frac{35}{128}\frac{d^2}{D^2} + 0.21\frac{d^3}{D^3}\right)\frac{3}{8}\sqrt{\frac{D}{d}}$$




$$K_I = \frac{4P}{B}\sqrt{\frac{\pi}{W}}\left\{1.6\left(\frac{a}{W}\right)^{1/2} - 2.6\left(\frac{a}{W}\right)^{3/2} + 12.3\left(\frac{a}{W}\right)^{5/2} - 21.2\left(\frac{a}{W}\right)^{7/2} + 21.8\left(\frac{a}{W}\right)^{9/2}\right\}$$




$$K_I = \frac{P}{B}\sqrt{\frac{\pi}{W}}\left\{16.7\left(\frac{a}{W}\right)^{1/2} - 104.7\left(\frac{a}{W}\right)^{3/2} + 369.9\left(\frac{a}{W}\right)^{5/2} - 573.8\left(\frac{a}{W}\right)^{7/2} + 360.5\left(\frac{a}{W}\right)^{9/2}\right\}$$

NAF
March 2010