Version NAF/4

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 30 April 2024 9.30 to 11.10

Module 3C9

FRACTURE MECHANICS OF MATERIALS AND STRUCTURES

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C9 Fracture Mechanics of Materials and Structures datasheet (8 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Detail the main factors that dictate the toughness of a brittle ceramic such as silica glass, and of a steel that contains inclusions. [40%]

(b) A uniform beam, of length ℓ, square cross-section $b \times b$ and second moment of area I of cross-section, contains an edge crack of length a at its mid-length, see Fig. 1. The beam is made from a linear elastic solid, of Young's modulus E. The ends of the beam are subjected to a moment M such that the ends of the beam rotate through a relative angle θ, where $\theta = CM$ and the compliance C is given by

$$C = \frac{\ell}{EI} + \frac{24ab}{EI(b-a)}.$$

(i) State the elastic strain energy ϕ of the beam for a prescribed value of $\theta = \theta_0$, and the potential energy P of the beam when the end moment is prescribed, such that $M = M_0$. [10%]

(ii) Hence obtain expressions for the energy release rate $\mathcal{G}$ and stress intensity factor K as a function of crack length a when the end rotation is prescribed, $\theta = \theta_0$. [20%]

(iii) Show explicitly that the expressions for $\mathcal{G}$ and K are unchanged when the end moment is prescribed, $M = M_0$. [10%]

(iv) Determine whether crack growth is stable or unstable, for a prescribed end moment, $M = M_0$. [20%]

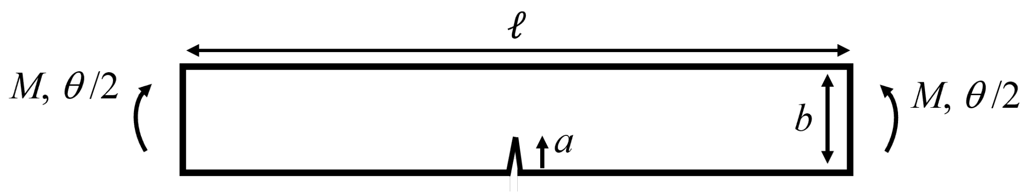

Fig. 1

2 (a) Explain why metallic alloys display an R-curve, such that the applied stress intensity factor K must be increased to give a crack extension Δa. Contrast this with the R-curve for a fibre-reinforced ceramic. [40%]

(b) A large steel sheet contains a central crack of length $2a_0$, and is subjected to a remote tensile stress σ^∞. The steel exhibits an R-curve behaviour such that the crack extension Δa scales with the stress intensity factor K according to:

$$
\begin{aligned}
K &\leq K_0\,, \quad \Delta a = 0 \\
&= K_0 + (K_1 - K_0)\sin\left(\frac{\pi}{2}\frac{\Delta a}{\lambda}\right)\,, \quad 0 < \Delta a < \lambda \\
&= K_1\,, \quad \Delta a \geq \lambda.
\end{aligned}
$$

(i) Assuming that linear elastic fracture mechanics prevails, obtain an expression for the value of σ^∞ that initiates crack growth, and explain why initial crack growth is stable. [10%]

(ii) With suitable use of sketches, explain the relation between Δa and σ^∞, and the effect of initial crack length upon the range of σ^∞ over which stable crack growth occurs. [20%]

(iii) Sketch the curve of σ^∞ versus extension of the ends of the sheet, with crack advance allowed for, in a displacement-controlled test. [20%]

(iv) Explain how the R-curve can be used to determine the $J - \Delta a$ resistance curve when linear elastic fracture mechanics is violated. [10%]

3 (a) A steel sheet contains a central hole and is loaded in cyclic compression-compression such that a region in the vicinity of the hole undergoes cyclic yield. Explain whether a fatigue crack will initiate and grow. [40%]

(b) A long steel tie-bar, that forms part of a truck suspension, is subjected to pulsating tension-tension random loading. The bar is of square cross-section $b \times b$ and contains a surface thumbnail crack of semi-circular shape and depth a_0 at mid-length of the bar, as sketched in Fig. 2. The tensile axial stress on the cross-section has a range $\Delta\sigma$, with probability density function in terms of a constant parameter σ_L given by:

$$\begin{aligned} p(\Delta\sigma) &= 1/\sigma_L\,, \quad 0 < \Delta\sigma < \sigma_L \\ &= 0\,, \quad \sigma > \sigma_L. \end{aligned}$$

(i) Obtain the relation between stress intensity factor K for a semi-circular crack of depth a and the remote stress σ by making use of the 3C9 datasheet, and assuming that $a/b << 1$. [10%]

(ii) Show that the Miner's law of linear summation of damage implies that the average crack growth rate per cycle of random loading is given by:

$$\langle \frac{\mathrm{d}a}{\mathrm{d}N} \rangle = \int_0^\infty \frac{\mathrm{d}a}{\mathrm{d}N} p(\Delta\sigma)\, d\Delta\sigma$$

where the crack growth rate $\mathrm{d}a/\mathrm{d}N$ is related to the stress intensity range ΔK by the usual Paris law:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = A\Delta K^n$$

in terms of the material constants A and n. [10%]

(iii) Hence obtain an expression for the number of cycles N_T required to double the depth of the semi-circular crack from its initial depth a_0. [30%]

(iv) Comment on the accuracy of Miner's law when the random loading is modified to include occasional large overloads. [10%]

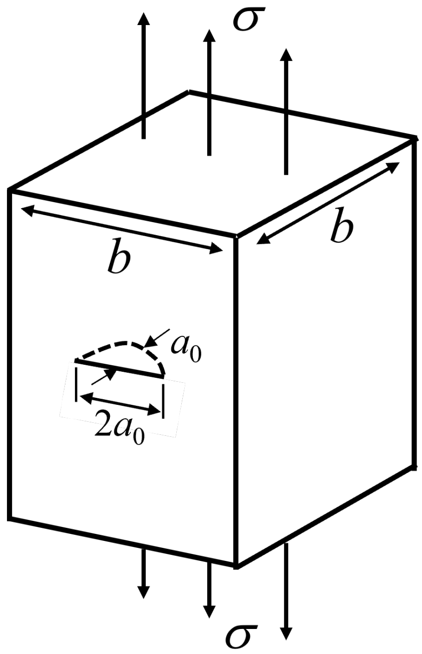

Fig. 2

4 (a) Explain why a sticking plaster, placed over an accidental flesh wound due to a knife stab, reduces the likelihood of a cut in the skin from advancing. [25%]

(b) A fatigue crack in a metallic plate is subjected to cyclic loading with stress intensity factor ranging from K_{min} to K_{max}. Explain why the crack growth rate scales with the cyclic rather than monotonic plastic zone size. [25%]

(c) Explain why the fracture toughness of a stack of thin sheets that are adhesively bonded together commonly exceeds that of a thick sheet of an aluminium alloy. [25%]

(d) Explain why linear elastic fracture mechanics can be used to characterise the failure of a metallic structure, with reference to small scale yielding. [25%]

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ENGINEERING TRIPOS PART IIA

Module 3C9 – FRACTURE MECHANICS OF MATERIALS AND STRUCTURES

DATASHEET

Crack tip plastic zone sizes

$$\text{diameter, } d_p = \begin{cases} \dfrac{1}{\pi}\left(\dfrac{K_I}{\sigma_y}\right)^2 & \text{Plane stress} \\ \dfrac{1}{3\pi}\left(\dfrac{K_I}{\sigma_y}\right)^2 & \text{Plane strain} \end{cases}$$

Crack opening displacement

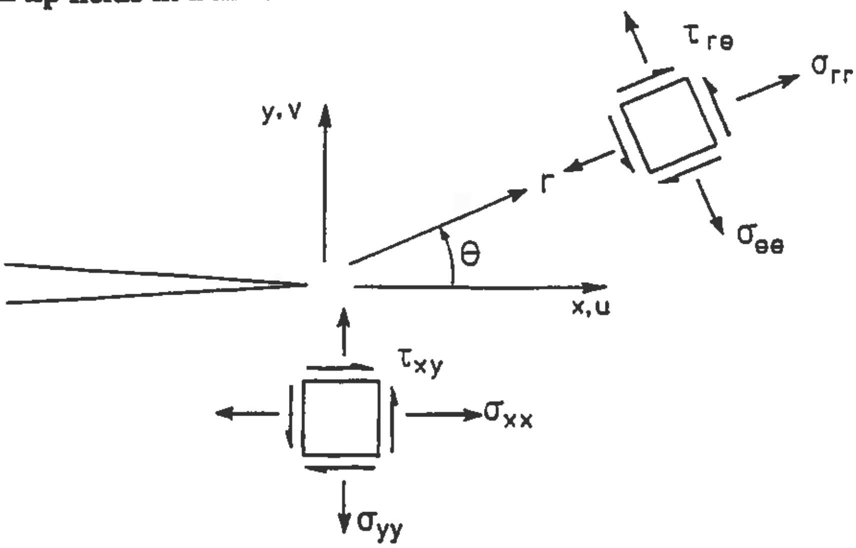
$$\delta = \begin{cases} \dfrac{K_I^2}{\sigma_y E} & \text{Plane stress} \\ \dfrac{1}{2}\dfrac{K_I^2}{\sigma_y E} & \text{Plane strain} \end{cases}$$

Energy release rate

$$G = \begin{cases} \dfrac{1}{E}K_I^2 & \text{Plane stress} \\ \dfrac{1-\nu^2}{E}K_I^2 & \text{Plane strain} \end{cases}$$

Related to compliance C: $G = \dfrac{1}{2}\dfrac{P^2}{B}\dfrac{dC}{da}$

Asymptotic crack tip fields in a linear elastic solid



Mode I

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$

$$u = \begin{cases} \dfrac{K_I}{G}\sqrt{\dfrac{r}{2\pi}}\left(\dfrac{1-\nu}{1+\nu} + \sin^2\dfrac{\theta}{2}\right)\cos\dfrac{\theta}{2} & \text{Plane stress} \\ \dfrac{K_I}{G}\sqrt{\dfrac{r}{2\pi}}\left(1 - 2\nu + \sin^2\dfrac{\theta}{2}\right)\cos\dfrac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$v = \begin{cases} \dfrac{K_I}{G}\sqrt{\dfrac{r}{2\pi}}\left(\dfrac{2}{1+\nu} - \cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2} & \text{Plane stress} \\ \dfrac{K_I}{G}\sqrt{\dfrac{r}{2\pi}}\left(2 - 2\nu - \cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$w = 0$$

Crack tip stress fields (cont'd)

Mode II

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\frac{3\theta}{2}$$

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = -\frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

$$u = \begin{cases} \dfrac{K_{II}}{G}\sqrt{\dfrac{r}{2\pi}}\left(\dfrac{2}{1+\nu} + \cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2} & \text{Plane stress} \\ \dfrac{K_{II}}{G}\sqrt{\dfrac{r}{2\pi}}\left(2 - 2\nu + \cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$v = \begin{cases} \dfrac{K_{II}}{G}\sqrt{\dfrac{r}{2\pi}}\left(\dfrac{\nu - 1}{1+\nu} + \sin^2\dfrac{\theta}{2}\right)\cos\dfrac{\theta}{2} & \text{Plane stress} \\ \dfrac{K_{II}}{G}\sqrt{\dfrac{r}{2\pi}}\left(-1 + 2\nu + \sin^2\dfrac{\theta}{2}\right)\cos\dfrac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$w = 0$$

Mode III

$$\tau_{zx} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}$$

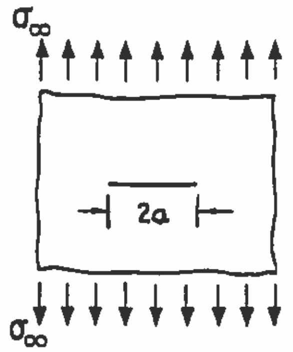
$$\tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

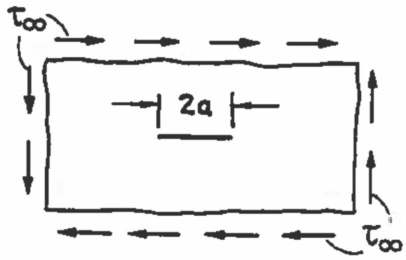
$$w = \frac{K_{III}}{G}\sqrt{\frac{2r}{\pi}} \sin\frac{\theta}{2}$$

$$u = v = 0$$

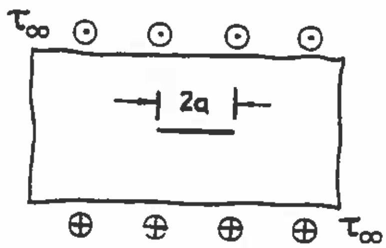
Tables of stress intensity factors



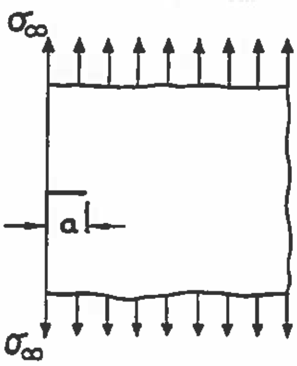
$$K_I = \sigma_\infty \sqrt{\pi a}$$



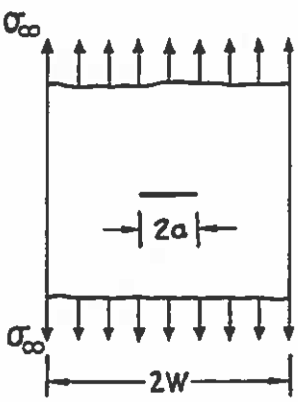
$$K_{II} = \tau_\infty \sqrt{\pi a}$$



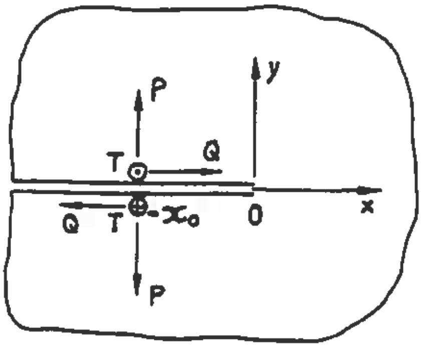
$$K_{III} = \tau_\infty \sqrt{\pi a}$$



$$K_I = 1.12\,\sigma_\infty \sqrt{\pi a}$$



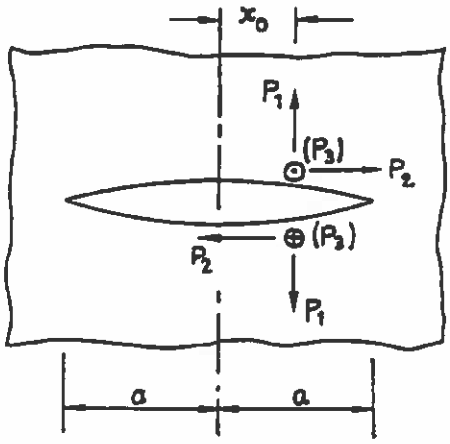
$$K_I = \sigma_\infty \sqrt{\pi a}\left(\frac{1 - a/2W + 0.326a^2/W^2}{\sqrt{1 - a/W}}\right)$$



$$K_I = \frac{2P}{\sqrt{2\pi x_o}}$$

$$K_{II} = \frac{2Q}{\sqrt{2\pi x_o}}$$

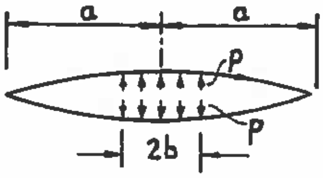
$$K_{III} = \frac{2T}{\sqrt{2\pi x_o}}$$



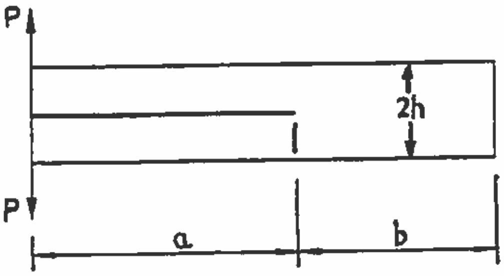
$$K_I = \frac{P_1}{\sqrt{\pi a}}\sqrt{\frac{a+x_o}{a-x_o}}$$

$$K_{II} = \frac{P_2}{\sqrt{\pi a}}\sqrt{\frac{a+x_o}{a-x_o}}$$

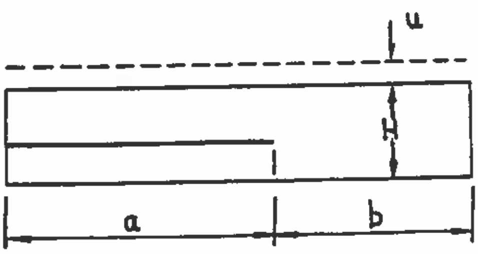
$$K_{III} = \frac{P_3}{\sqrt{\pi a}}\sqrt{\frac{a+x_o}{a-x_o}}$$



$$K_I = \frac{2pb}{\sqrt{\pi a}}\frac{a}{b}\arcsin\frac{b}{a}$$

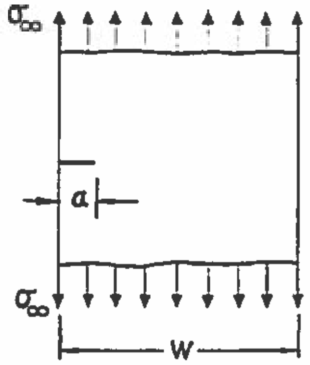


$$K_I = \frac{2\sqrt{3}}{h\sqrt{h}}\frac{Pa}{B} \qquad h \ll a \text{ and } h \ll b$$



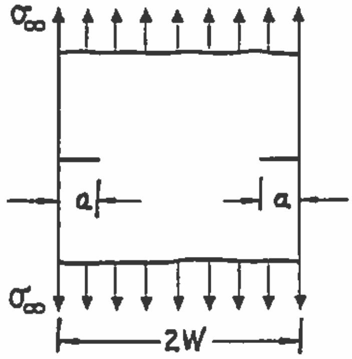
$$K_I = \sqrt{\frac{1}{2\alpha H}}\,Eu \qquad H \ll a \text{ and } H \ll b$$

$$\alpha = \begin{cases} 1-\nu^2 & \text{Plane stress} \\ 1-3\nu^2-2\nu^3 & \text{Plane strain} \end{cases}$$

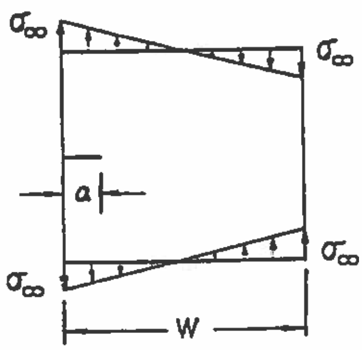


$a/W < 0.7$

$$K_I = \sigma_\infty\sqrt{\pi a}\left(1.12 - 0.23\frac{a}{W} + 10.6\frac{a^2}{W^2} - 21.7\frac{a^3}{W^3} + 30.4\frac{a^4}{W^4}\right)$$

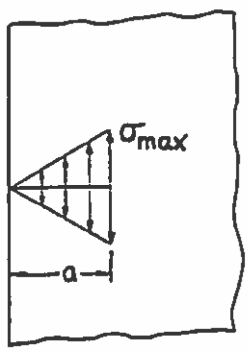


$$K_I = \sigma_\infty\sqrt{\pi a}\left(\frac{1.12 - 0.61a/W + 0.13a^3/W^3}{\sqrt{1-a/W}}\right)$$

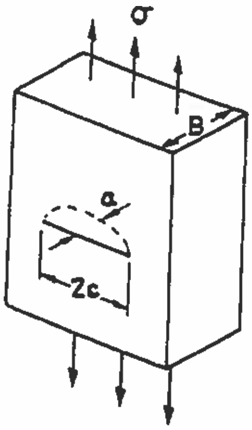


$a/W < 0.7$

$$K_I = \sigma_\infty\sqrt{\pi a}\left(1.12 - 1.39\frac{a}{W} + 7.3\frac{a^2}{W^2} - 13\frac{a^3}{W^3} + 14\frac{a^4}{W^4}\right)$$

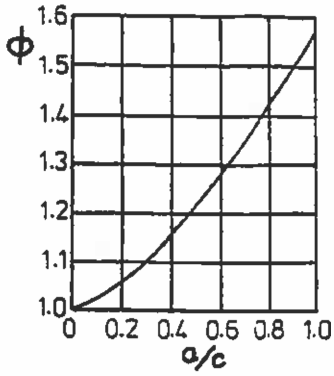


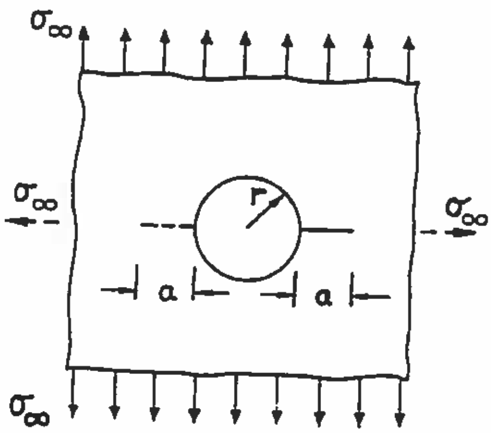
$$K_I = 0.683\,\sigma_{max}\sqrt{\pi a}$$



$$K_I = \frac{1.12}{\Phi}\sigma\sqrt{\pi a}$$

$$\Phi = \int_0^{\pi/2}\left(1 - \frac{c^2 - a^2}{c^2}\sin^2\theta\right)^{\frac{1}{2}} d\theta$$



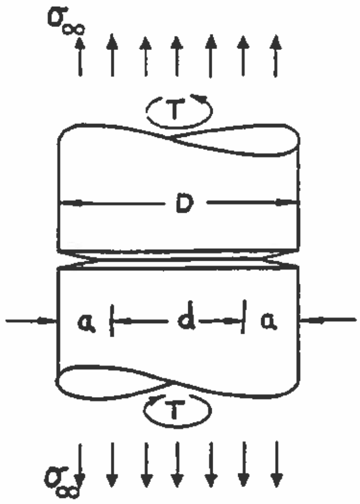


$$K_I = \sigma_\infty\sqrt{\pi a}\; F\left(\frac{a}{r}\right)$$

value of $F(a/r)$†

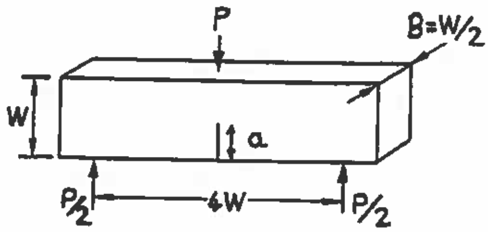
$\frac{a}{r}$	One crack U	One crack B	Two cracks U	Two cracks B
0.00	3.36	2.24	3.36	2.24
0.10	2.73	1.98	2.73	1.98
0.20	2.30	1.82	2.41	1.83
0.30	2.04	1.67	2.15	1.70
0.40	1.86	1.58	1.96	1.61
0.50	1.73	1.49	1.83	1.57
0.60	1.64	1.42	1.71	1.52
0.80	1.47	1.32	1.58	1.43
1.0	1.37	1.22	1.45	1.38
1.5	1.18	1.06	1.29	1.26
2.0	1.06	1.01	1.21	1.20
3.0	0.94	0.93	1.14	1.13
5.0	0.81	0.81	1.07	1.06
10.0	0.75	0.75	1.03	1.03
∞	0.707	0.707	1.00	1.00

†U = uniaxial σ_∞ B = biaxial σ_∞.

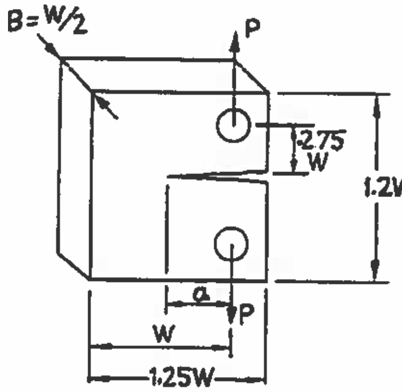


$$K_I = \sigma_\infty \sqrt{\pi a}\left(\frac{D}{d}+\frac{1}{2}+\frac{3}{8}\frac{d}{D}-0.36\frac{d^2}{D^2}+0.73\frac{d^3}{D^3}\right)\frac{1}{2}\sqrt{\frac{D}{d}}$$

$$K_{III} = \frac{16T}{\pi D^3}\sqrt{\pi a}\left(\frac{D^2}{d^2}+\frac{1}{2}\frac{D}{d}+\frac{3}{8}+\frac{5}{16}\frac{d}{D}+\frac{35}{128}\frac{d^2}{D^2}+0.21\frac{d^3}{D^3}\right)\frac{3}{8}\sqrt{\frac{D}{d}}$$



$$K_I = \frac{4P}{B}\sqrt{\frac{\pi}{W}}\left\{1.6\left(\frac{a}{W}\right)^{1/2}-2.6\left(\frac{a}{W}\right)^{3/2}+12.3\left(\frac{a}{W}\right)^{5/2}-21.2\left(\frac{a}{W}\right)^{7/2}+21.8\left(\frac{a}{W}\right)^{9/2}\right\}$$



$$K_I = \frac{P}{B}\sqrt{\frac{\pi}{W}}\left\{16.7\left(\frac{a}{W}\right)^{1/2}-104.7\left(\frac{a}{W}\right)^{3/2}+369.9\left(\frac{a}{W}\right)^{5/2}-573.8\left(\frac{a}{W}\right)^{7/2}+360.5\left(\frac{a}{W}\right)^{9/2}\right\}$$

NAF
March 2010