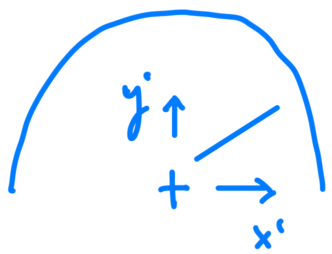
Q1a)

i) Thin-walled open section

$$J \approx \frac{1}{3} t^3 \left(a + \frac{2\pi a}{4}\right) \approx 0.857\, a t^3$$

ii) Semicircle:



$$I_{x'x'} = t \int_0^{\pi} (a \sin\alpha)^2 a\, d\alpha = t a^3 \frac{\pi}{2}$$

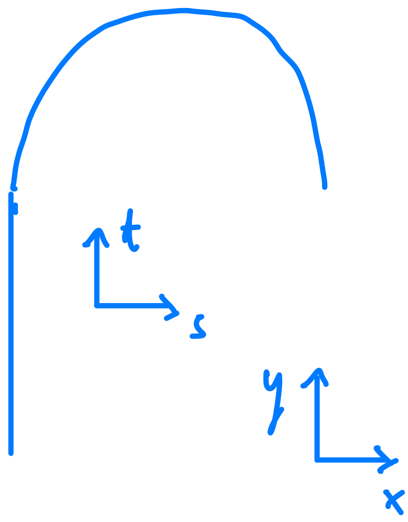
$$I_{y'y'} = t \int_0^{\pi} (a \cos\alpha)^2 a\, d\alpha = t a^3 \frac{\pi}{2}$$

Centre of gravity:

$$y'_s\, t a \pi = t \int_0^{\pi} a \sin\alpha\, a\, d\alpha \quad \Rightarrow y'_s = \frac{2a}{\pi}$$

$$I_{x'_s x'_s} = t a^3 \frac{\pi}{2} - t \pi a \left(\frac{2a}{\pi}\right)^2 = 0.298\, a^3 t$$

Entire section:



$$A = t(\pi a + a) \approx 4.14\, a t$$

$$x_s = \frac{1}{4.14at}\left(-at\, 2a - \pi a t\, a\right)$$

$$\approx -1.24\, a$$

$$y_s = \frac{1}{4.14at}\left(at \frac{a}{2} + \pi a t\, 1.64a\right)$$

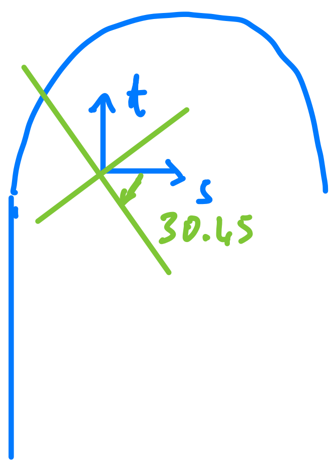
$$\approx 1.36 a$$

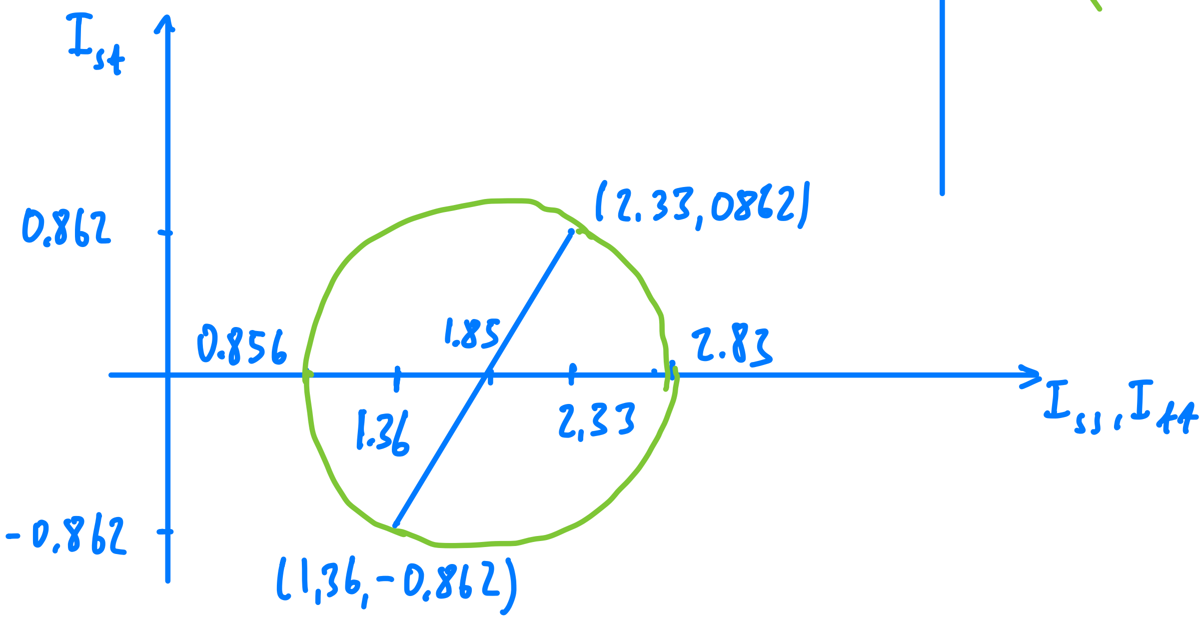
$$I_{ss} = \frac{t a^3}{12} + a t (0.86a)^2 + 0.298 a^3 t + \pi a t \cdot (0.277a)^2 = 1.36 a^3 t$$

$$I_{tt} = a t \cdot (0.76a)^2 + t a^3 \frac{\pi}{2} + \pi a t \cdot (0.24a)^2 = 2.33 a^3 t$$

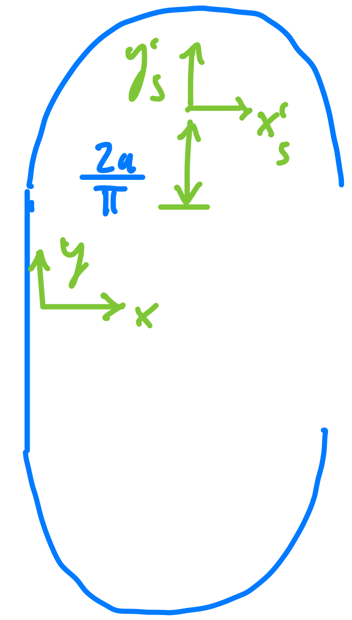
$$I_{ts} = a t(-0.76a)(-0.86a) + \pi a t (0.24a)(0.277a) = 0.862 a^3 t$$

$$I = \begin{pmatrix} 1.36 & 0.862 \\ 0.862 & 2.33 \end{pmatrix} a^3 t$$





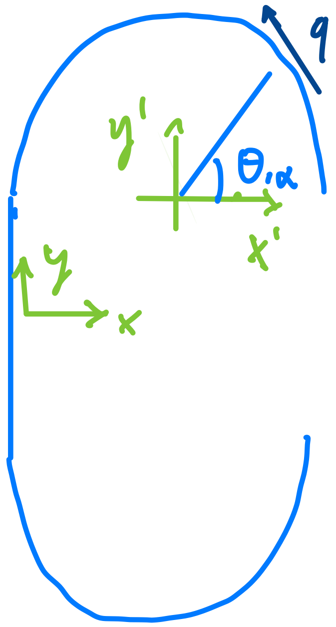
b)



Semicircle : $I_{x_s'x_s'} = 0.298a^3t$

Entire Section:

$$I_{xx} = \frac{ta^3}{12} + 2\left(0.298a^3t + \left(\frac{2a}{\pi} + \frac{a}{2}\right)^2 \pi a t\right) = 8.80a^3t$$



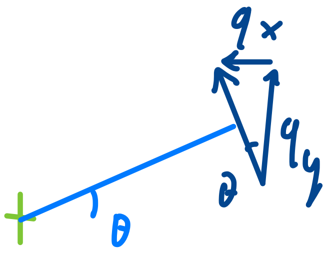
First moment of area

$$\bar{y}A = \int_0^\theta t\left(\frac{a}{2} + a\sin\alpha\right)a\,d\alpha$$

$$= \frac{1}{2}a^2t(2 + \theta - 2\cos\theta)$$

Shear flow (in semicircle)

$$q = \frac{F\bar{y}A}{I_{xx}}$$



$$q_x = \frac{F\bar{y}A}{I_{xx}}\sin\theta \qquad q_y = \frac{F\bar{y}A}{I_{xx}}\cos\theta$$

Force and moment resultant at the origin $x'y'$

$$H = \int_0^{\pi} q_x \, a \, d\theta = \frac{F}{I_{xx}} \int_0^{\pi} \frac{1}{2} a^2 t (2 + \theta - 2\cos\theta) \sin\theta \, a \, d\theta$$

$$= \frac{F a^2 t}{2 I_{xx}} a (4 + \pi) = 0.406 F$$

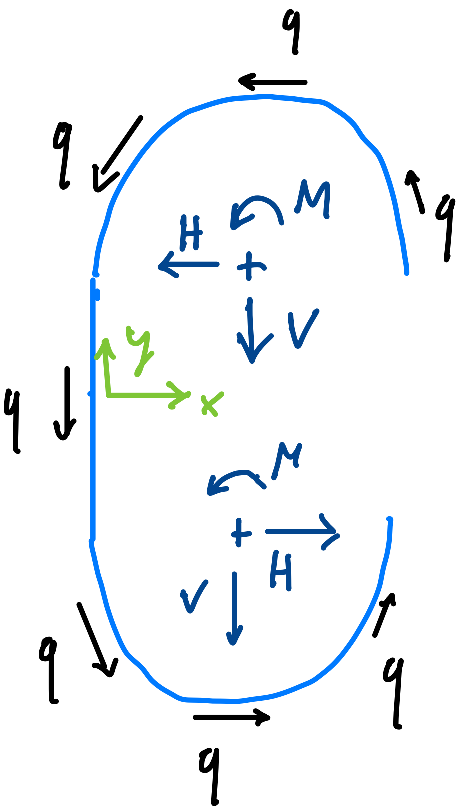
$$V = \int_0^{\pi} q_y \, a \, d\theta = \frac{F}{I_{xx}} \int_0^{\pi} \frac{1}{2} a^2 t (2 + \theta - 2\cos\theta) \cos\theta \, a \, d\theta$$

$$= - \frac{F a^2 t}{2 I_{xx}} a (2 + \pi) = 0.292 F$$

$$M = \int_0^{\pi} q \, a^2 \, d\theta = \frac{F}{I_{xx}} \int_0^{\pi} a^4 t (2 + \theta - 2\cos\theta) \, d\theta$$

$$= \frac{F a^4 t}{I_{xx}} \frac{1}{2} \pi (4 + \pi) = 1.27 Fa$$

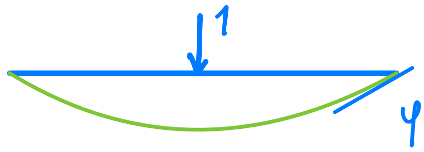
Moments about the origin of xy



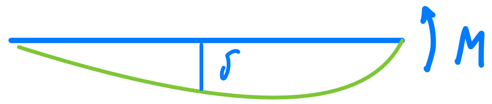
$$F \cdot d = 2 \cdot 1.27\, Fa + 0.606\, Fa - 2 \cdot a\, 0.292 F$$

$$\Rightarrow d = 2.36\, a$$

Q2a) Databook cases

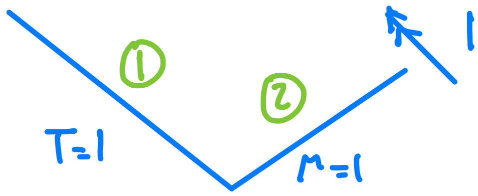


$$y = \frac{l^2}{16EI}$$



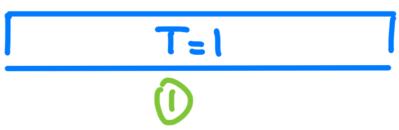
$$\delta = M\frac{l^2}{16EI}$$

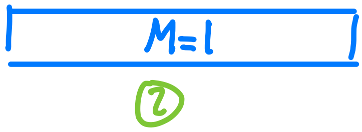
b) iii)



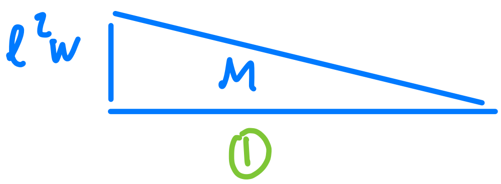
The bending moment at A is chosen as the single redundancy.

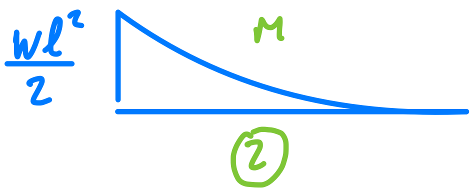
Due to unit moment:

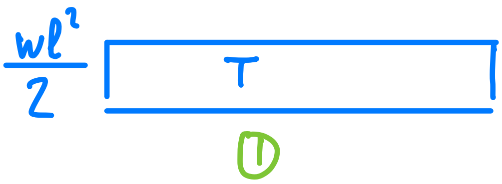




Due to w







Rotation due to unit moment:

$$1 \cdot \beta_v = \frac{1}{EI}\int M_v^2 \, ds + \frac{1}{GJ}\int T_v^2 \, ds$$

$$= \frac{l}{EI} + \frac{l}{GJ} = \frac{3l}{2EI}$$

Rotation due to w

$$1\cdot\beta_a = \frac{1}{EI}\int_0^{\ell} 1\cdot\frac{ws^2}{2}\,ds + \frac{1}{GJ}\int_0^{\ell} 1\cdot\frac{w\ell^2}{2}\,ds$$

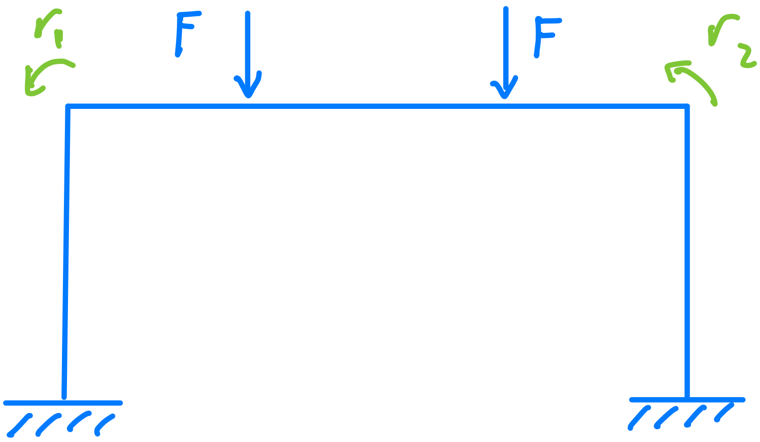
$$= \frac{1}{EI}\,\frac{w\ell^3}{6} + \frac{1}{2EI}\,\frac{w\ell^3}{2} = \frac{5}{12}\,\frac{w\ell^3}{EI}$$

Compatibility

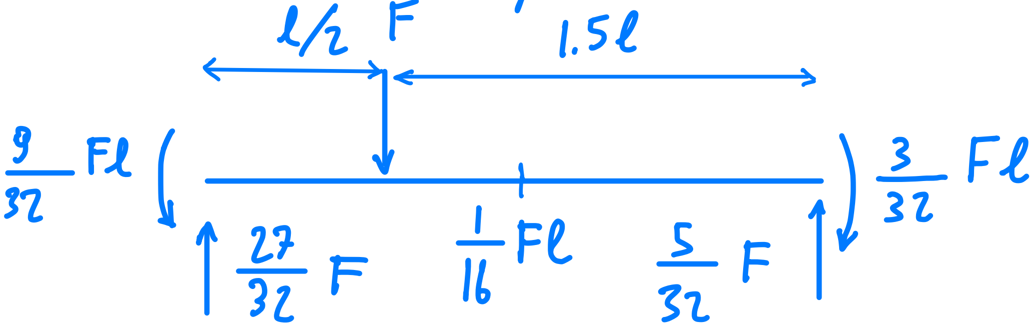
$$M_A\,\frac{3\ell}{2}EI - \frac{5}{12}\,\frac{w\ell^3}{EI} = 0$$

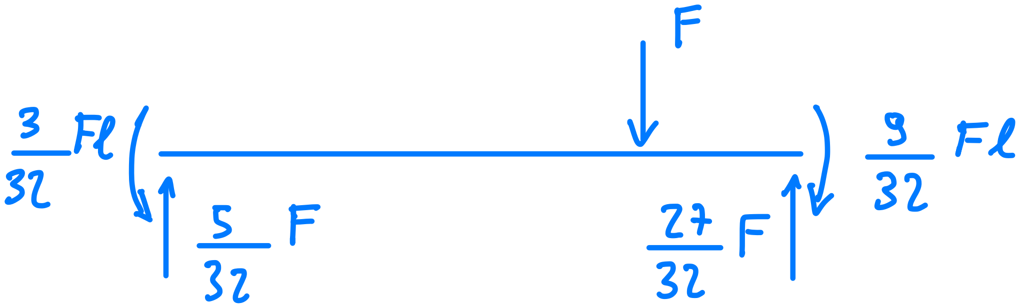
$$\Longrightarrow M_A = \frac{5\ell^2 w}{18}$$

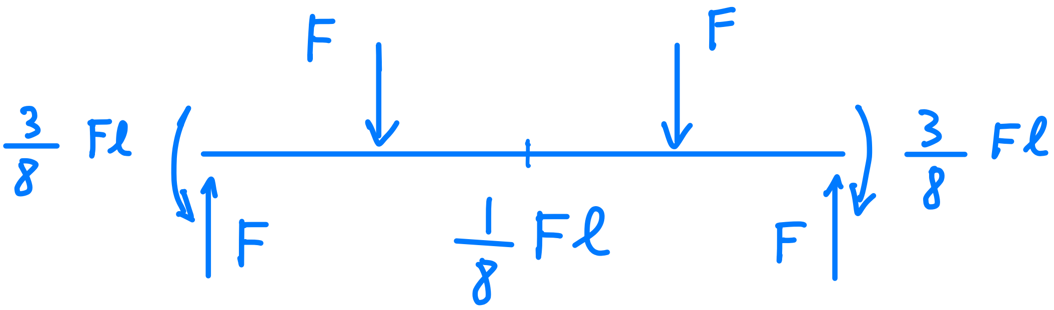
ii)



Constrained system (Data Book)







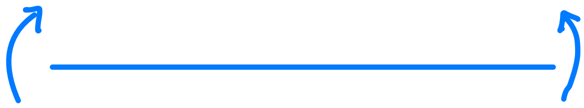
Stiffness matrix (by inspection)

$$EI\begin{pmatrix} \frac{4}{l}+\frac{4}{2l} & \frac{2}{2l} \\ \frac{2}{2l} & \frac{4}{l}+\frac{4}{2l} \end{pmatrix} = \frac{EI}{l}\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$$

$$\frac{EI}{l}\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = -\begin{pmatrix} \frac{3}{8}Fl \\ \frac{3}{8}Fl \end{pmatrix}$$

$$\Rightarrow \quad r_1 = r_2 = -\frac{3Fl^2}{56EI}$$

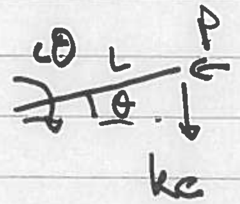
Supperposition of system with only r_1 and r_2



$$\frac{4EI}{\ell} r_1 + \frac{2EI}{\ell} r_2 = \frac{9}{28} F\ell \qquad \frac{4EI}{\ell} r_1 + \frac{2EI}{\ell} r_2 = \frac{9}{28} F\ell$$

Supperposition of system with r_1, r_2 and F

- Bending moment at A: $\frac{1}{8} F\ell + \frac{9}{28} F\ell = \frac{25}{56} F\ell$

Q3 (a.i)  $e = L\sin\theta$ $\quad c\theta + ke\cdot L\cos\theta - PL\sin\theta = 0$

$$\frac{c\theta}{L\sin\theta} + \frac{kL^2\sin\theta\cos\theta}{L\sin\theta} = P.$$

$$\Rightarrow \quad P = \frac{c}{L}\cdot\frac{\theta}{\sin\theta} + kL\cos\theta \simeq \underline{\underline{\frac{c}{L} + kL}}\ ; \ \theta \text{ small}$$

(a.ii) if θ expanded: $\sin\theta \simeq \theta - \theta^3/6$: $\cos\theta \simeq 1 - \theta^2/2$.

$$\Rightarrow \quad P = \frac{c}{L}\cdot\frac{\theta}{\theta[1-\theta^2/6]} + kL\left(1 - \theta^2/2\right) \qquad [1+x]^n \simeq 1 + nx.$$

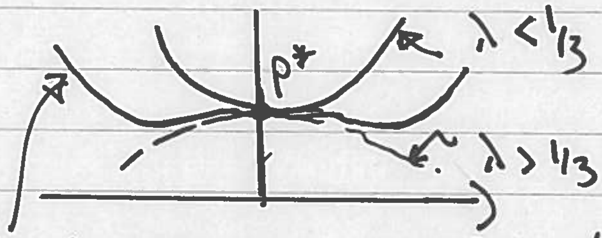
$$\simeq \frac{c}{L}\left[1 + \theta^2/6\right] + kL\left(1 - \theta^2/2\right)$$

$$P \simeq \frac{c}{L} + kL + \theta^2\underbrace{\left[\frac{c}{6L} - \frac{kL}{2}\right]}\quad \text{if} \quad \frac{c}{6L} - \frac{kL}{2} = 0$$

$$\Rightarrow \quad \underline{kL_0 = 2c/6L = 1/3\,(c/L)}$$

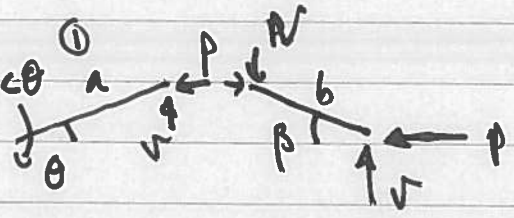
(a.iii) Sketch exact variation: $P = \frac{c}{L}\frac{\theta}{\sin\theta} + kL\cos\theta$.

Let $\lambda = kL^2/c$: when $\lambda = 1/3$, switch in dirn at $\theta = 0$



$$P = \underbrace{\frac{c}{L} + kL}_{P^*} + \theta^2\cdot\frac{c}{6L}\left[1 - 3\lambda\right]$$

when λ just $> 1/3$: there is a downward dip before rising again.

b). 

① $c\theta - aV - a\theta P = 0$

$Pa\theta = c\theta - aV$

② $\tan\beta \simeq \beta = V/P$ for axial eqlm

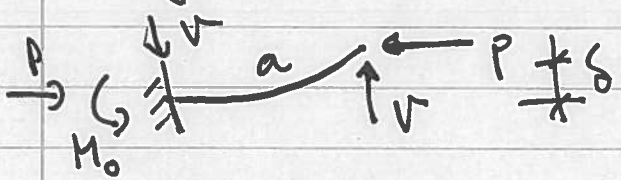
and $a\theta = b\beta$ for continuity of pins .

$$\Rightarrow P a\theta = c\theta - a(P\beta) \quad a\theta/b. \quad \Rightarrow P a\theta = c\theta - a.P.a\theta/b$$

$$aP + Pa^2/b = c \quad \Rightarrow P[ab + a^2] = cb.$$

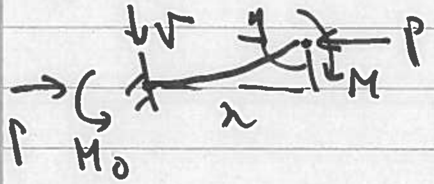
$$\Rightarrow P = \frac{cb}{a(a+b)}$$

(c) Key here is to note that $\tan\beta = V/P$. for axial force



$$M_0 + Va + P\delta = 0$$

$$M_0 = -P\delta - aV \quad - (A).$$

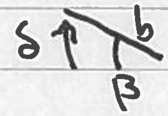


$$M - M_0 - Py - xV = 0. \leftarrow \text{eqlm eqn}$$

$$(A) \quad -M_0 - xV = -M + Py$$

$$\Rightarrow +P\delta + aV - xV = +EIy'' + Py \quad : \quad y'' = d^2y/dx^2.$$

But $\tan\beta = V/P$ $\quad \tan\beta = \delta/b$



$$\Rightarrow V = P\tan\beta = P\delta/b : \quad \Rightarrow +P\delta + \frac{P\delta}{b}[a - x] = EIy'' + Py$$

$$\underbrace{P/EI}_{\alpha^2}\left[\delta/b[a - x] + \delta\right] = y'' + \alpha^2 y \qquad \alpha^2 = P/EI$$

$$\alpha^2 . \underbrace{\delta\left[a/b + 1 - x/b\right]}_{C.F.} = y'' + \alpha^2 y$$

$$\Rightarrow \text{General soln:} \quad y = \overbrace{A\cos\alpha x + B\sin\alpha x}^{\text{constants}} + \delta\left[\frac{a}{b} + 1\right] - \frac{\delta x}{b}.$$

$$y' = -\alpha A\sin\alpha x + \alpha B\cos\alpha x - \delta/b$$

$$y'' = -\alpha^2 A\cos\alpha x - \alpha^2 B\sin\alpha x$$

P.T.O.

$x = 0, \; y = 0 \Rightarrow \quad 0 = A + \delta\left[a/b + 1\right]$

$\Rightarrow A = -\delta\left[a/b + 1\right]$

$x = 0, \; y' = 0 \Rightarrow \quad 0 = \alpha B - \delta/b \Rightarrow \quad B = \delta/b\alpha$

At $x = a$, $M = 0$ at pin

$\Rightarrow \quad 0 = -\cancel{\alpha^2} A \cos\alpha a - \cancel{\alpha^2} B \sin\alpha a$

$\Rightarrow \tan\alpha a = -A/B = +\cancel{\delta}\left[\frac{a}{b} + 1\right] \cdot \frac{b\alpha}{\cancel{\delta}}$

$\Rightarrow \tan\alpha a = b\alpha\left[a/b + 1\right] = \alpha\left[a + b\right]$

i.e. $\tan\alpha a = \alpha\left[a + b\right]$

(1)

4a) S.E. bending = $\frac{1}{2} EI \int_0^L \kappa^2 \, dx$ where $\kappa^2 = (w'')^2$

The end shortening = $\underbrace{\frac{1}{2} \int w'^2 \, dx}$ ⇒ W.D by $P = P \times (\Delta)$

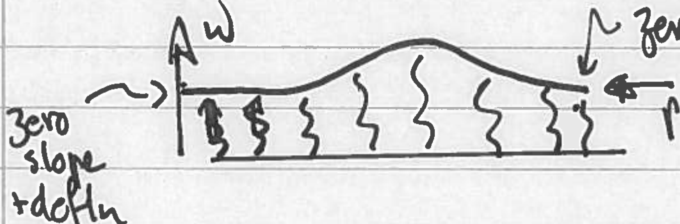
for springs/unit length, S.E = $\frac{1}{2} k w^2$ ∴ integrate for full length.

b) Let $w = a_0 + a_1 (x/L) + a_2 (x/L)^2 + a_3 (x/L)^3 + a_4 (x/L)^4$

Define $\xi = x/L$ ∴ $w = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + a_4 \xi^4$.

$$\partial w / \partial \xi = a_1 + 2 a_2 \xi + 3 a_3 \xi^2 + 4 a_4 \xi^3$$

$$\partial^2 w / \partial \xi^2 = 2 a_2 + 6 a_3 \xi + 12 a_4 \xi^2 .$$



$w = 0$ at $\xi = 0$ ⇒ $a_0 = 0$
$w' = 0$ at $\xi = 0$ ⇒ $a_1 = 0$

⇒ $w = a_2 \xi^2 + a_3 \xi^3 + a_4 \xi^4$.

$w = 0$ at $\xi = 1$ ⇒ $\underline{a_2 + a_3 + a_4 = 0}$

$w'' = 0$ at $\xi = 1$ ⇒ $\underline{\underline{2 a_2 + 6 a_3 + 12 a_4}} = 0$

Let a_2 be independent ⇒ $a_4 = -(a_2 + a_3)$

⇒ $a_2 + 3 a_3 + 6 \underbrace{\left[-(a_2 + a_3) \right]}_{a_4} = 0$

PTO

$a_2 + 3a_3 - 6a_2 - 6a_3 = 0 \quad -3a_3 = 5a_2 \Rightarrow a_3 = -\frac{5a_2}{3}$

$\Rightarrow \quad w = a_2\xi^2 - \frac{5a_2}{3}\xi^3 + \frac{2a_2}{3}\xi^4$

$a_4 = -a_2 - a_3$
$= -a_2 + \frac{5a_2}{3}$
$= \frac{2a_2}{3}$

$dw/d\xi = 2a_2\xi - 5a_2\xi^2 + \frac{8a_2}{3}\xi^3$

$d^2w/d\xi^2 = 2a_2 - 10a_2\xi + 8a_2\xi^2$

If $\xi = x/L \Rightarrow d\xi = dx/L$: $\frac{dw}{d\xi} = L\frac{dw}{dx} \Rightarrow \frac{dw}{dx} = \frac{1}{L}\frac{dw}{d\xi}$

$\Rightarrow d^2w/dx^2 = \frac{1}{L^2}\cdot\frac{d^2w}{d\xi^2}$

S.E. bending $= \frac{1}{2}EI\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{1}{2}EI\int_0^1\left(\frac{1}{L^2}\frac{d^2w}{d\xi^2}\right)^2 L\,d\xi$

$= \frac{1}{2}\frac{EI}{L^3}\int_0^1 (d^2w/d\xi^2)^2\, d\xi = \frac{EI}{2L^3}\int_0^1 [2a_2 - 10a_2\xi + 8a_2\xi^2]^2 d\xi$

$\int[2 - 10\xi + 8\xi^2]^2 d\xi = \int_0^1 [4 + 100\xi^2 + 64\xi^4 - 40\xi + 32\xi^2 - 160\xi^3]\, d\xi$

$= \left[4\xi + 100\xi^3/3 + 64\xi^5/5 - 40\xi^2/2 + 32\xi^3/3 - \frac{160\xi^4}{4}\right]_0^1$

$= 4 + 100/3 + 64/5 - 20 + 32/3 - 40 = 4/15$

Also need $\int_0^1 (w')^2 d\xi = a_2^2\int [2\xi - 5\xi^2 + 8/3\,\xi^3]^2 d\xi$

$= a_2^2\int [4\xi^2 + 25\xi^4 + \frac{64}{9}\xi^6 - 20\xi^3 + \frac{32}{3}\xi^4 - \frac{80}{3}\xi^5]\, d\xi$

$= a_2^2\left[\frac{4\xi^3}{3} + \frac{25\xi^5}{5} + \frac{64\xi^7}{9\times7} - \frac{20\xi^4}{4} + \frac{32}{3}\cdot\xi^5/5 - \frac{80}{3}\cdot\xi^6/6\right]_0^1$

$$= a_2^2\left[4/3 + \cancel{8} + \frac{64}{63} - \cancel{8} + 32/\cancel{15} - \frac{80}{18}\right] = 4/105\, a_2^2$$

$$\int_0^1 w^2 d\xi \Rightarrow \int a_2^2\left[\xi^2 - 5/3\,\xi^3 + 2/3\,\xi^4\right]^2 d\xi$$

$$= a_2^2 \int\left[\xi^4 + \frac{25}{9}\xi^6 + \frac{4}{9}\cdot\xi^8 - \frac{10}{3}\xi^5 + \frac{4}{3}\xi^6 - \frac{20}{9}\xi^7\right] d\xi$$

$$= a_2^2\left[\xi^5/5 + \frac{25}{9}\cdot\frac{\xi^7}{7} + \frac{4}{9}\xi^9/9 - \frac{10}{3}\xi^6/6 + \frac{4}{3}\xi^7/7 - \frac{20}{9}\xi^8/8\right]_0^1$$

$$= a_2^2\left[1/5 + \frac{25}{63} + \frac{4}{81} - \frac{10}{18} + \frac{4}{21} - \frac{20}{72}\right] = \frac{19}{5670}a_2^2.$$

$$\Rightarrow \tfrac{1}{2}EI\int_0^L w''^2 dx = \frac{EI}{2L^3}\int_0^1 \left(d^2w/d\xi^2\right)^2 d\xi = \frac{EIa_2^2}{2L^3}\cdot\frac{4}{5}$$

$$\tfrac{1}{2}P\int_0^L (w')^2 dx = \frac{P}{2L}\int_0^1 \left(\frac{dw}{dx}\right)^2 d\xi = \underline{\underline{\frac{P}{2L}\cdot a_2^2\cdot\frac{4}{105}}}$$

$$\tfrac{1}{2}k\int_0^L w^2 dx = \tfrac{1}{2}kL\int_0^1 w^2 d\xi = \underline{\underline{\tfrac{1}{2}kLa_2^2\cdot\frac{19}{5670}}}$$

$$\Pi = -\frac{Pa_2^2}{L}\cdot\frac{2}{105} + \frac{2}{5}\cdot\frac{EIa_2^2}{L^3} + kLa_2^2\frac{19}{2\times5670}$$

$$\partial\pi/\partial(a_2^2) = 0 \Rightarrow P = \frac{EI}{L^2}\cdot\frac{\cancel{2}}{\cancel{5}}\frac{\cancel{105}^{21}}{\cancel{2}} + kL^2\cdot\frac{19}{\cancel{2\times5670}_{54}}\cdot\frac{\cancel{105}}{2}.$$

$$\Rightarrow P = 21\frac{EI}{L^2} + \underbrace{\frac{19}{216}}_{0.088}\cdot kL^2$$