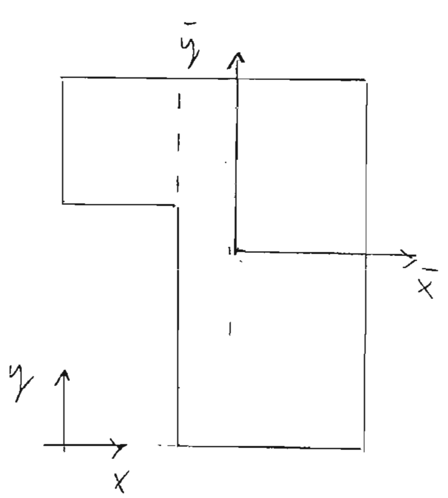
Qu 1)

a)



$$A = 0{,}6 \cdot 1{,}2\,\ell^2 + 0{,}4^2\,\ell^2 = 0{,}88\,\ell^2$$

$$x_S = \frac{0{,}6 \cdot 1{,}2 \cdot 0{,}7\,\ell^3 + 0{,}4^2 \cdot 0{,}2\,\ell^3}{0{,}88\,\ell^2} = 0{,}609\,\ell$$

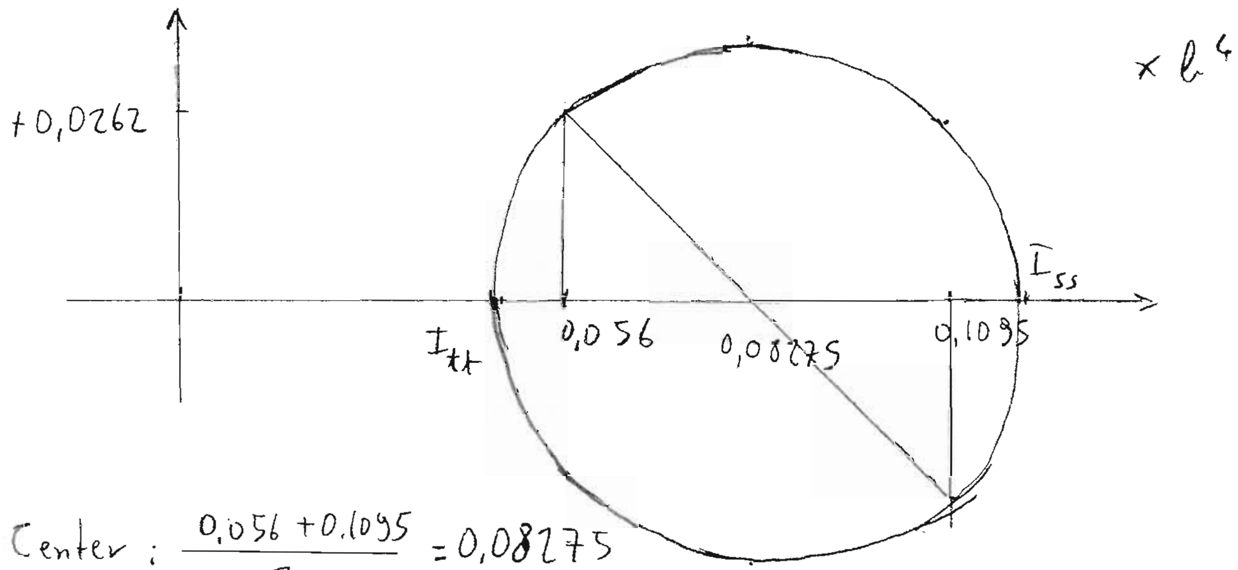
$$y_S = \frac{0{,}6 \cdot 1{,}2 \cdot 0{,}6\,\ell^3 + 0{,}4^2 \cdot 1{,}0\,\ell^3}{0{,}88\,\ell^2} = 0{,}673\,\ell$$

$$I_{\bar{x}\bar{x}} = \frac{0{,}4^3 \cdot 0{,}4\,\ell^4}{12} + 0{,}4^2 \cdot 0{,}327^2\,\ell^4 + \frac{1{,}2^3 \cdot 0{,}6}{12} + 0{,}6 \cdot 1{,}2 \cdot 0{,}073^2\,\ell^4 = \underline{0{,}1095\,\ell^4}$$

$$I_{\bar{y}\bar{y}} = \frac{0{,}4^3 \cdot 0{,}4\,\ell^4}{12} + 0{,}4^2 \cdot 0{,}409^2\,\ell^4 + \frac{0{,}6^3 \cdot 1{,}2\,\ell^4}{12} + 0{,}6 \cdot 1{,}2 \cdot 0{,}091^2\,\ell^4 = \underline{0{,}05646\,\ell^4}$$

$$I_{\bar{x}\bar{y}} = 0 + 0{,}4^2 \cdot 0{,}327(-0{,}409)\,\ell^4 + 0 + 0{,}6 \cdot 1{,}2 \cdot (-0{,}073) \cdot 0{,}091\,\ell^4 = \underline{-0{,}02618\,\ell^4}$$

Mohr's circle

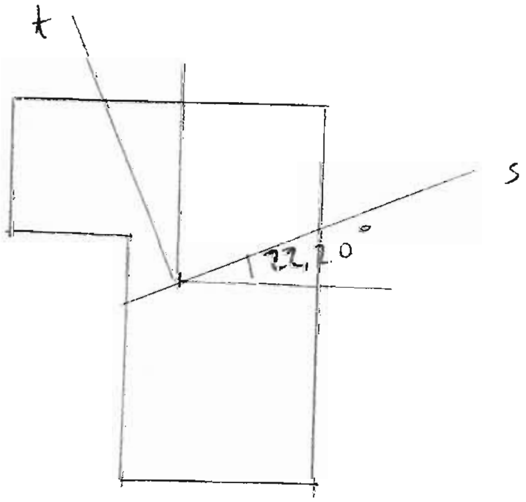


Center : $\dfrac{0{,}056 + 0{,}1095}{2} = 0{,}08275$

Radius : $\sqrt{(0{,}1095 - 0{,}08275)^2 + 0{,}0262^2} = 0{,}03376$

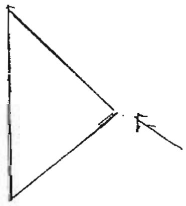
$I_{ss} = 0{,}08275 + 0{,}0376 = 0{,}12035$

$I_{tt} = 0{,}08275 - 0{,}0376 = 0{,}04515$



b)

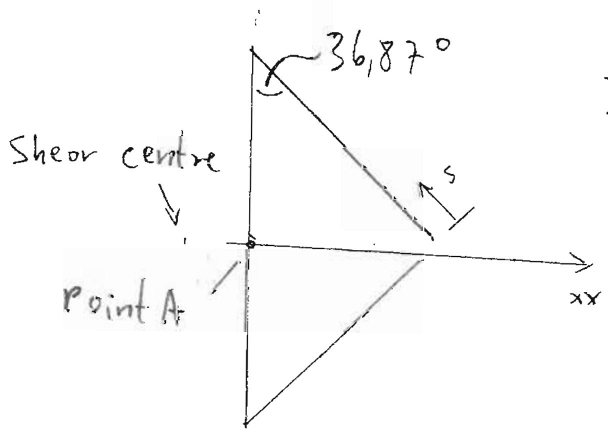
i) Warping torsion important because thin-walled <u>open</u> cross-section with shear centre outside the cross-section. Torsional stiffness could be increased by:



Weld here and turn into a closed cross-section

Alternatively the torsional stiffness can be increased by adding longitudinal and/or transversal stiffeners.

ii)



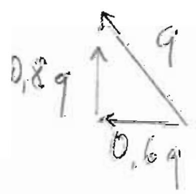
$$I_{xx} = \frac{8^3 \cdot t \cdot b}{12} + 2\cdot\left[t\int_0^{5b}(0{,}8s)^2\,ds\right]$$

$$= 42{,}67\,t b^3 + 53{,}33\,t b^3 = 96{,}0\,t b^3$$

Shear flow in the "flange" $\qquad q = \dfrac{Q\bar{y}A}{I_{xx}}$

$$q = \frac{Q \cdot st \cdot 0{,}8\,s/2}{96{,}0\,t b^3} = 2{,}083\cdot 10^{-3}\,Q\,\frac{s^2}{b^3}$$

Moment around point A



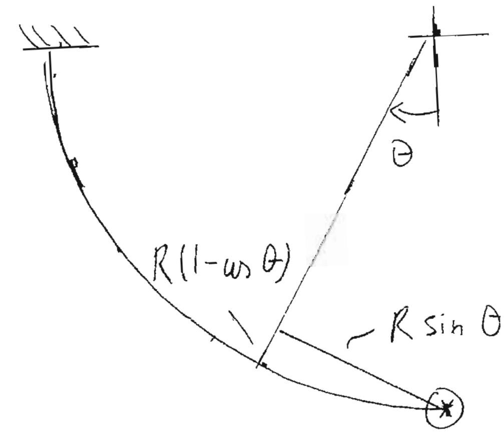
$$M = 2\cdot\int_0^{5b}\left(0{,}6q\cdot 0{,}8s + 0{,}8q(3b-0{,}6s)\right)ds$$

$$= 2\int_0^{b} 2{,}4\,qb = 0{,}01358\,Q\int_0^{5b}\frac{s^2}{b^2}\,ds = 0{,}5658\,bQ$$

$$\Rightarrow x_s = -0{,}5658\,b$$

Qu 2

a)



$M = R \sin\theta$

$T = R(1-\cos\theta)$

Tip displacement for the system with unit load:

$$\delta_{11} = \int_0^{\pi/2} \frac{R^2 \sin^2\theta}{EI} R\,d\theta + \int_0^{\pi/2} \frac{R^2(1-\cos\theta)^2}{GJ} R\,d\theta + \frac{1}{EA}1{,}2R$$

$$EI\,\delta_{11} = \int_0^{\pi/2} \left(R^3 \sin^2\theta + \frac{R^3}{3}\left(1 - 2\cos\theta + \cos^2\theta\right)\right) d\theta + \frac{1{,}2R^3}{10}$$

$$= R^3\frac{\pi}{4} + R^3\frac{\pi}{6} - \frac{R^3}{3}\pi + \frac{R^3\pi}{12} + \frac{1{,}2R^3}{10}$$

$$= \frac{\pi}{6}R^3 + \frac{1{,}2R^3}{10}$$

Tip displacement for the system with F

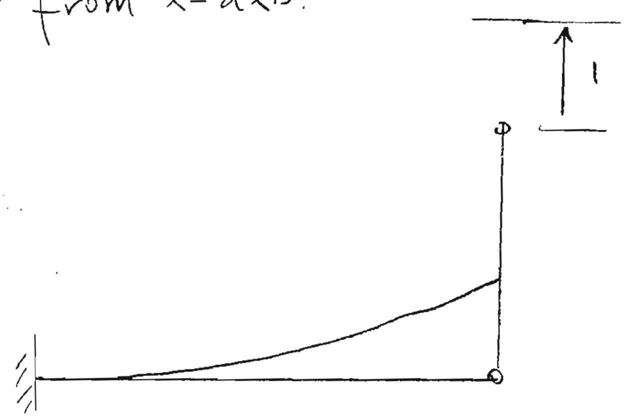
$$\delta_{10} = -\int_0^{\pi/2} \frac{R^2 \sin^2\theta\, F}{EI} R\,d\theta - \int_0^{\pi/2} \frac{R^2(1-\cos\theta)^2}{GJ} F R\,d\theta$$

$$\Longrightarrow EI\,\delta_{10} = -\frac{\pi}{6}R^3F$$

Cable force: $X = \frac{\delta_{10}}{\delta_{11}} = 0{,}8135F$

b) To obtain the influence line for reaction force need to apply a unit displacement.

View from x-axis:



According to a) a cable force of "1" leads to a displacement of δ_{11}. Hence the force corresponding to the unit displacement is

$$R = \frac{1}{\frac{\pi R^3}{6} + \frac{1{,}2R^3}{10}} \approx 1{,}5537$$

Need to compute the corresponding beam displacement.

$$EI\,w'' = M = 1{,}5537\,R \sin\theta$$

$$EI\,w' = -1{,}5537\,R\cos\theta + c_1$$

$$EI\,w = -1{,}5537\,R\sin\theta + c_1\theta + c_2$$

$$w'\left(\tfrac{\pi}{2}\right) = 0 \Rightarrow c_1 = 0$$

$$w\left(\tfrac{\pi}{2}\right) = 0 \Rightarrow c_2 = 1{,}5537R$$

$$\Rightarrow EI\,w = -1{,}5537R\sin\theta + 1{,}5537R$$

According to a) for a perfect structure one has

$$X \delta_{11} + \delta_{10} = 0$$

For a structure with fabrication error

$$X \delta_{11} + \delta_{10} = + \frac{0,1}{100} 1,2 \cdot R$$

$$\Rightarrow \quad X = \frac{1}{\delta_{11}} \left(+ \frac{0,1}{100} 1,2 \cdot R + \delta_{10} \right)$$

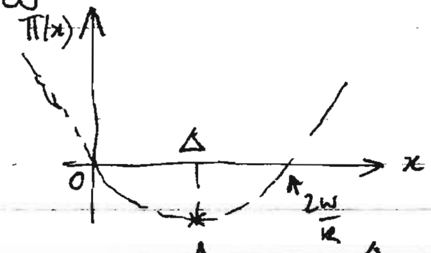
$$= \frac{EI}{0,6436 R^3} \left(+ 1,2 \cdot 10^{-3} R + \frac{0,5236 R^3 F}{EI} \right)$$

$$= 0,8135 F + 1,865 \cdot 10^{-3} \frac{EI}{R^2}$$

3D4 Structural Analysis and Stability
Engineering Tripos Part IIA 2014

Q3 a) i) Total Potential Energy $\Pi(x) = \frac{1}{2}kx^2 - Wx$

$$= \frac{1}{2}kx\left(x - \frac{2W}{k}\right)$$



min at $\Delta = W/k$ Equilibrium

Total PE at equilib $= \frac{1}{2}k\left(\frac{W}{k}\right)\left(\frac{W}{k} - \frac{2W}{k}\right) = -\frac{1}{2}W\left(\frac{W}{k}\right) = -\frac{1}{2}W\Delta$

b) ii) Work done $= \int \text{force} \times \text{dist} = \int_0^{\Delta} kx \, dx = \frac{1}{2}k\Delta^2$

$$= \frac{1}{2}k\Delta . \Delta = \underline{\frac{1}{2}W\Delta}$$

iii) Answer to ii) does not hold.

e.g. spring force $= cx^n$

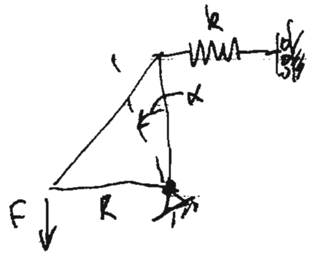
W.D. $= \int_0^{\Delta} (cx^n)\,dx = \frac{1}{(n+1)} c\Delta^{n+1}$

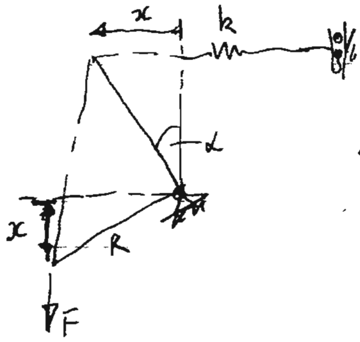
At equilib $W = c\Delta^n$

so Work done $= \frac{1}{n+1}(c\Delta^n).\Delta = \frac{1}{(n+1)} W.\Delta$

which only equals answer in ii) if $n = 1$ (linear spring).

c b) i)





$x = R\sin\alpha$

$$\Pi = \tfrac{1}{2}kx^2 - Fx \quad = \tfrac{1}{2}kR^2\sin^2\alpha - F.R\sin\alpha$$

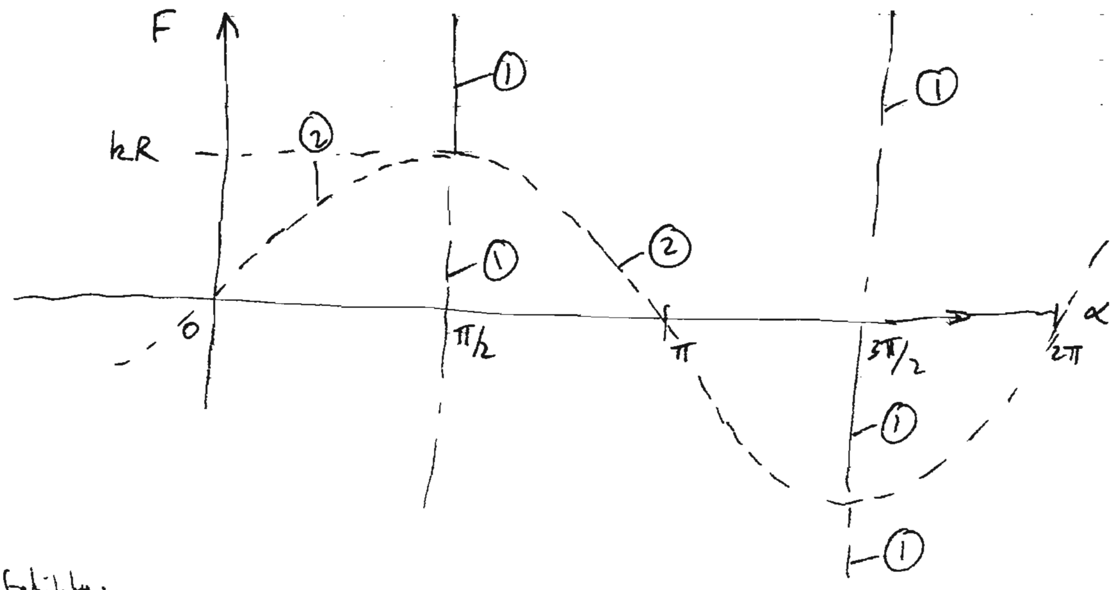
Equilib. when $\frac{d\Pi}{d\alpha}=0$ $\quad = \quad kR^2\cos\alpha\sin\alpha - FR\cos\alpha$

$$= R.\cos\alpha\,(kR\sin\alpha - F)$$

which $=0$ when either $\cos\alpha = 0$ ①
or $kR\sin\alpha - F = 0$ ②

① $\rightarrow$ $\alpha = \pi/2,\ 3\pi/2,$ etc

② $\rightarrow$ $F = kR\sin\alpha$



Stability:
Along ① $\quad \frac{d^2\Pi}{d\alpha^2} = ?$ $\quad \frac{d\Pi}{d\alpha} = R\left(\tfrac{1}{2}kR\sin 2\alpha - F\cos\alpha\right)$

$$\frac{d^2\Pi}{d\alpha^2} = R\left(kR\cos 2\alpha + F\sin\alpha\right)$$

when $\alpha = \pi/2$ $\quad \frac{d^2\Pi}{d\alpha^2} = R\left(kR\cos\pi + F\sin\pi/2\right) = R(-kR+F)$

which is +ve when $F > kR$. (stable)

$\alpha = 3\pi/2$ $\quad \frac{d^2\Pi}{d\alpha^2} = R\left(kR\cos 3\pi + F\sin 3\pi/2\right) = R(-kR-F)$

which is +ve when $F < -kR$

C b) i) cont'd.

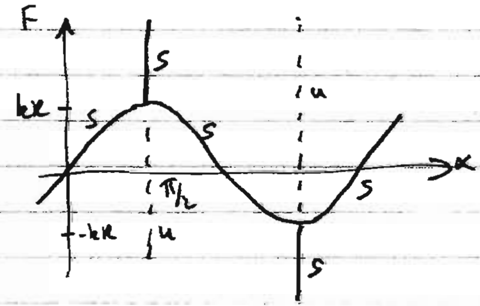
Along ②
$$\frac{d^2\Pi}{dR^2} = R\left(kR\cos 2\alpha + F\sin\alpha\right)$$

with $kR\sin\alpha = F$ on ②

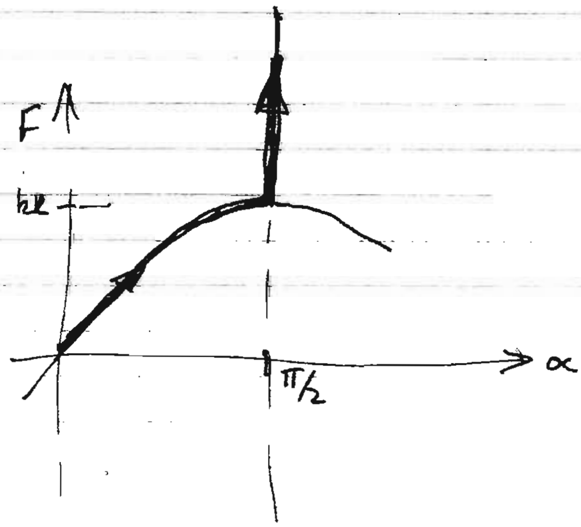
so
$$\frac{d^2\Pi}{dR^2} = R\left(kR\left[1 - 2\sin^2\alpha\right] + \left(kR\sin\alpha\right)\sin\alpha\right]$$
$$= R(kR)\left[1 - 2\sin^2\alpha + \sin^2\alpha\right)$$
$$= kR^2\left[1 - \sin^2\alpha\right]$$
$$= kR^2\left[\cos^2\alpha\right] = \text{always +ve}$$

(except at $\alpha = \pi/2, 3\pi/2$, etc).

so path ② is stable everywhere (except where it hits path ①)



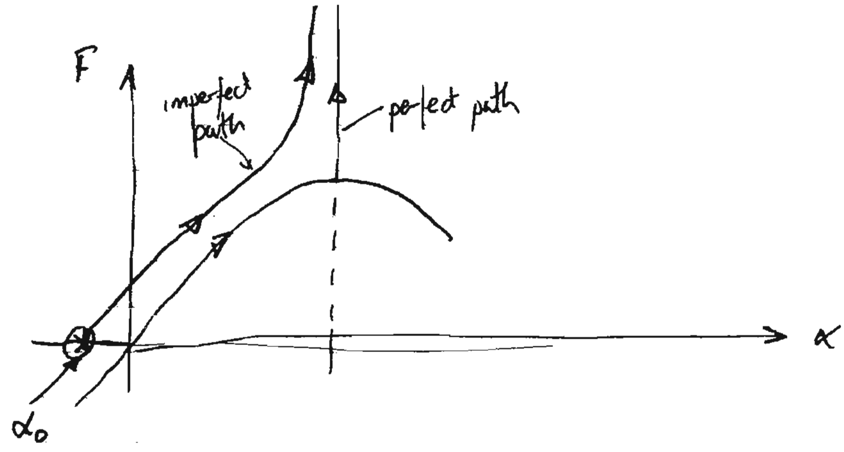
so when loaded from F=0



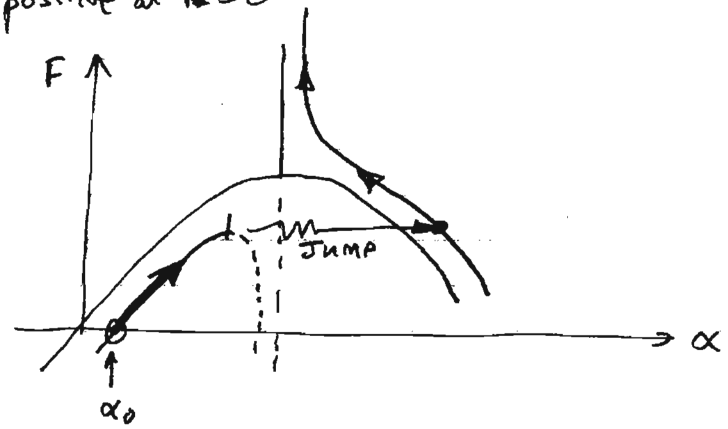
ie. as F rises from 0 to kR the panel tips over to increasing angles of α.

At F = kR the panel has point A pointing vertically downwards, and under increasing load it maintains this position.

c ~~b~~) ii) α_0 negative at $F=0$:



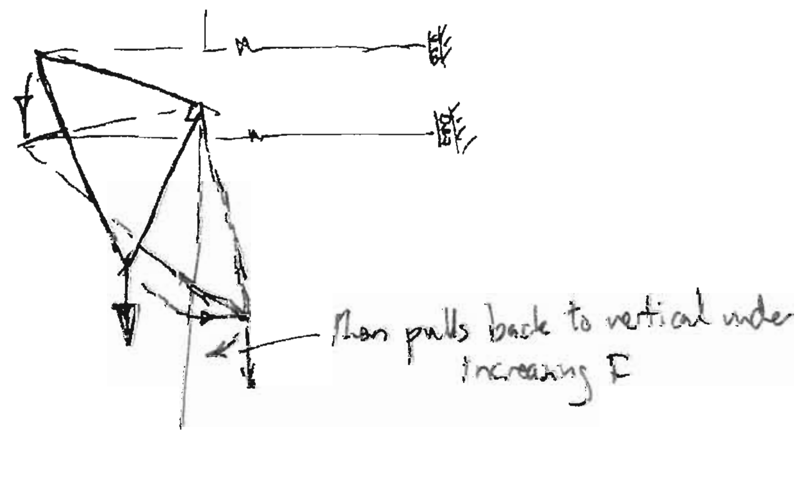
α_0 positive at $F=0$



In first case (α_0 -ve) the panel tracks the path for the perfect case fairly closely. The α angle increases steadily and smoothly and asymptotes toward $\alpha = \pi/2$ as $F \to \infty$.

In the second case (α_0 +ve) the panel again tips downwards under increasing load, but just before it reaches $\alpha = \pi/2$ it suddenly jumps from a limit point across to a solution just past $\pi/2$. On further addition of load, this point pulls back asymptotically towards the vertical $\alpha = \pi/2$ state as $F \to \infty$

Sudden jump



3D4

Q4 a) $\begin{pmatrix} M_A \\ M_B \end{pmatrix} = k \begin{bmatrix} s+7 & sc \\ sc & s+6 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$ This being $k\begin{bmatrix} s & sc \\ sc & s \end{bmatrix}$ from column

and $(3+4)k$ at A

far end pinned, far end fixed

and $(3+3)k$ at B

both far ends pinned

$\begin{bmatrix} M_A \\ M_B \end{bmatrix} \rightarrow k\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ from AB

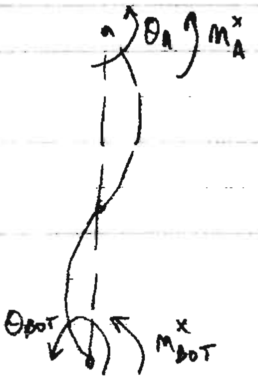
$+ \; k\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ from left beam

$+ \; k\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$ from right beam

$k\begin{bmatrix} 7 & 2 \\ 2 & 8 \end{bmatrix}$

+ column terms

For LH column $\begin{pmatrix} M_A^x \\ M_{BOT} \end{pmatrix} = k \begin{bmatrix} s & sc \\ sc & s \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_{BOT} \end{bmatrix}$ when considered in isolation



~~so 1st eqn~~ $\rightarrow \cancel{M_A^x = ks\theta_A}$

Now $M_{BOT}^x = 0$ (pin ended)

so $0 = ksc\,\theta_A + ks\,\theta_{BOT}$ (2nd eqn)

so $\theta_{BOT} = -c\theta_A$

Subst. into 1st eqn

$$M_A^x = ks\theta_A + ksc\,\theta_{BOT}$$
$$= ks\theta_A + ksc(-c\theta_A)$$
$$= ks(1-c^2)\theta_A$$

For RH column, similar, but $\theta_{BOT} = 0$ so immediately, from 1st eqn of s, c matrix

$M_B^x = ks\theta_B$

Total

$$\begin{pmatrix} M_A \\ M_B \end{pmatrix} = k\begin{bmatrix} s(1-c^2)+7 & 2 \\ 2 & s+8 \end{bmatrix}\begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$$

Q4

b) i) $k\begin{bmatrix} s+7 & sc \\ sc & s+6 \end{bmatrix}$

Buckles when $\det K = 0 \Rightarrow (s+7)(s+6) - (sc)^2 = 0$

$$(1-c^2)s^2 + 13s + 42 = 0$$

P/P_E	s	c	$(1-c^2)s^2$	$13s$	Σ
1.0	2.46	1	0	31.98	31.98
2.0	0.1428	24.6841	-12.4	1.86	-10.5
2.4	-1.3006	-3.3703	-17.52	-16.9	-34.4
2.6	-2.2490	-2.2312	-20.12	-29.24	-49.4
3.0	-5.0320	-1.4157	-25.4	-53.0	-78

$\leftarrow -42 \Rightarrow P/P_E = 2.5$

$$2.5 = \frac{P}{P_E} = \frac{\pi^2 EI/l^2}{\pi^2 EI/L^2} \qquad l^2 = \frac{L^2}{2.5} \qquad l = \frac{L}{\sqrt{2.5}} = 0.6235\,L$$

$= 0.6235\,(4)$
$= \underline{2.53\,m}$
effective length

$$P_{cr} = \pi^2 EI/l^2 = \frac{\pi^2 (210\times10^9\ N/m^2)(98610\times10^{-8}\ m^4)}{(2.53)^2\ m^2}$$

$= \underline{319.3\ MN}$

b) ii) When $P/P_E = 2.5$

$$s \approx \frac{-1.3006 - 2.2490}{2} \approx -1.7748$$

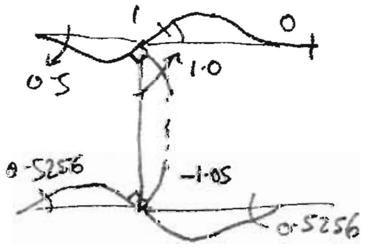
$$c \approx \frac{-3.3703 - 2.2312}{2} \approx -2.8008.$$

$$sc \approx 4.9708$$

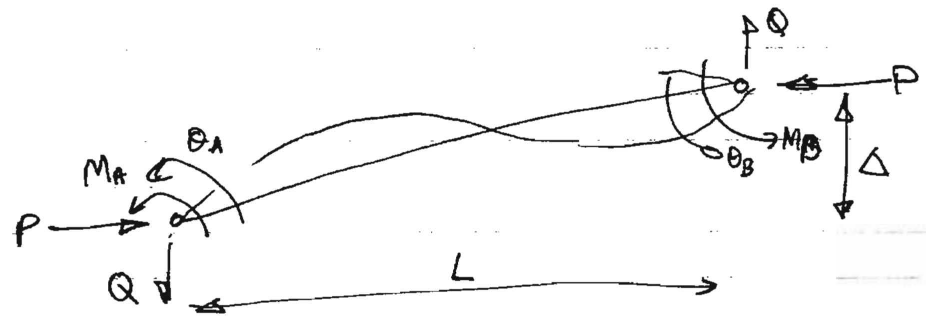
1st eqn $\quad M_A = 0 = k\left[\ \cancel{\ }\ 7 - 1.7748 \quad 4.9708\ \right]\begin{bmatrix}\theta_A \\ \theta_B\end{bmatrix}$

$\therefore\ 5.2252\,\theta_A + 4.9708\,\theta_B = 0$

so if $\theta_A = 1 \qquad \theta_B = \dfrac{-5.2252}{4.9708} = -1.0512$



4 c)



Let $\frac{\Delta}{L} = \gamma$

Slope deflection eqns

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = k \begin{bmatrix} s & sc \\ sc & s \end{bmatrix} \overbrace{\begin{bmatrix} \theta_A - \gamma \\ \theta_B - \gamma \end{bmatrix}}^{\text{new angles at end}}$$

$$= k \begin{bmatrix} s & sc & -s(1+c) \\ sc & s & -s(1+c) \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \gamma \end{bmatrix}$$

Moments about A $\circlearrowleft +$

$$M_A + M_B + QL + P\Delta = 0$$

$$QL = -(M_A + M_B) - P\Delta$$

$$M_A = k(s\theta_A + sc\theta_B - s(1+c)\gamma)$$

$$M_B = k(sc\theta_A + s\theta_B - s(1+c)\gamma)$$

$$-\Sigma \rightarrow = -k[s(1+c)\theta_A + s(1+c)\theta_B - 2s(1+c) \Delta/L]$$

$$QL = k \Big[\underbrace{-s(1+c) \quad -s(1+c)}_{\text{good, so symmetric matrix}} \quad + 2s(1+c) - \overbrace{\frac{PL}{k}}^{-P\Delta} \Big] \begin{bmatrix} \theta_A \\ \theta_B \\ \Delta/L \end{bmatrix}$$

whence

$$\underset{\text{shear.}}{\begin{bmatrix} M_A \\ M_B \\ QL \end{bmatrix}} = k \begin{bmatrix} s & sc & -s(1+c) \\ sc & s & -s(1+c) \\ -s(1+c) & -s(1+c) & 2s(1+c) - \frac{PL}{k} \end{bmatrix} \underset{\text{Sway deflection}}{\begin{bmatrix} \theta_A \\ \theta_B \\ \Delta/L \end{bmatrix}}$$

3x3 matrix req'd.