Q1)

From Databook:

$J = 378\, cm^4 \quad I_{xx} = 38750\, cm^4 \quad I_{yy} = 12570\, cm^4$

a)
$$J = 378 - \frac{1}{3}\, 7.5 \cdot 2.5^3 = 338.9$$

b) We can provide only an estimate. The restraint warping torsion is related to the bending of the flanges. The following provides an approx. lower bound

$$I_f = \frac{(31.12 - 7.5)^3 \cdot 2.5}{12} = 2745.4$$

$$I_{yy} \approx 2 \cdot 2745.4 = 5490.7$$

$$\Gamma = \frac{(32.7 - 2.5)^2}{4} \cdot 5490.7 = 12.52 \cdot 10^4$$

c) Cut piece properties

$A = 7.5 \cdot 2.5 = 18.75 \qquad x_s = 11.81 \quad y_s = 15.11$

$$I_{x'x'} = \frac{7.5 \cdot 2.5^3}{12} = 9.77$$

$$I_{y'y'} = \frac{7.5^3 \cdot 2.5}{12} = 87.9$$

Centroid of cross-section:

$$x_s'' = \frac{-18.75 \cdot 11.81}{201 - 18.75} = -1.22$$

$$y_s'' = \frac{-18.75 \cdot 15.11}{201 - 18.75} = -1.55$$

Second moments of entire cross-section

$$I_{x''x''} = 38750 + 1.55^2 \cdot 201 - 977 - 18.75 \cdot 16.66^2$$
$$= 34018$$

$$I_{y''y''} = 12570 + 1.22^2 \cdot 201 - 87.99 - 18.75 \cdot 13.03^2$$
$$= 9597$$

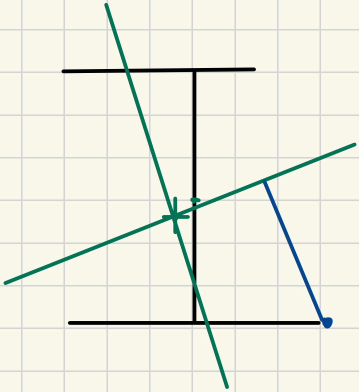
$$I_{x''y''} = 1.22 \cdot 1.55 \cdot 201 - 16.66 \cdot 13.03 \cdot 18.75$$
$$= -3690$$

By plotting the Mohr's circle or computing the eig vals & eig vecs we obstain

$$\begin{pmatrix} 34018 & -3690 \\ -3680 & 9597 \end{pmatrix} \Rightarrow \quad I_{ss} = 34563 \quad I_{tt} = 9052$$

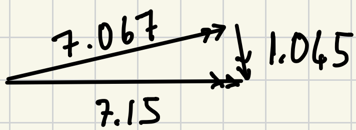
$$\theta = 8.41°$$

d)



Weight per meter $182.25 \cdot 7840 \cdot 10^{-6} \cdot 10 \approx 14.29\ \mathrm{N/cm}$

$$M = 14.29 \frac{L^2}{2} = 7.144\, L^2$$



Coordinates of point A $x_A'' = 16.78 \quad y_A'' = 14.81$

$$\begin{pmatrix} 0.9892 & -0.1463 \\ 0.1463 & 0.9892 \end{pmatrix} \begin{pmatrix} 16.78 \\ 14.81 \end{pmatrix} = \begin{pmatrix} 14.43 \\ 17.10 \end{pmatrix}$$

$$\sigma = \frac{7.067 L^2}{34563} \cdot 17.10 + \frac{1.045 L^2}{9052} 14.43 = 0.17 L^2$$

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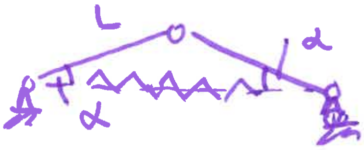
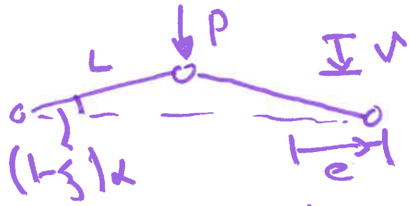
2 (ai)





ξ = fraction of rotation

lateral extension $= e = 2L\cos(1-\xi)\alpha - 2L\cos\alpha$.

vertical displacement $= v = L\sin\alpha - L\sin(1-\xi)\alpha$

2 (a.ii) Total potential energy; $U = \tfrac{1}{2}ke^2 - Pv$

$$U = \tfrac{1}{2}k\left[2L\cos(1-\xi)\alpha - 2L\cos\alpha\right]^2 - P\left[L\sin\alpha - L\sin(1-\xi)\alpha\right]$$

$$= 2kL^2\left[\cos(1-\xi)\alpha - \cos\alpha\right]^2 - PL\left[\sin\alpha - \sin(1-\xi)\alpha\right]$$

ξ = deformed variable $\Rightarrow \partial U/\partial\xi = 0 \equiv$ eqlm

$$\partial U/\partial\xi = 4kL^2\left[\cos(1-\xi)\alpha - \cos\alpha\right]\cdot -\sin(1-\xi)\alpha \cdot -\alpha$$
$$- PL\left[-\cos(1-\xi)\alpha\right]\cdot -\alpha = 0$$

$$\Rightarrow P\cancel{L}\cancel{\alpha}\cos(1-\xi)\alpha - 4kL^{\cancel{2}}\cancel{\alpha}\sin(1-\xi)\alpha\left[\cos(1-\xi)\alpha - \cos\alpha\right] = 0$$

$$\Rightarrow P = 4kL\left[\sin(1-\xi)\alpha - \cos\alpha \cdot \tan(1-\xi)\alpha\right]$$

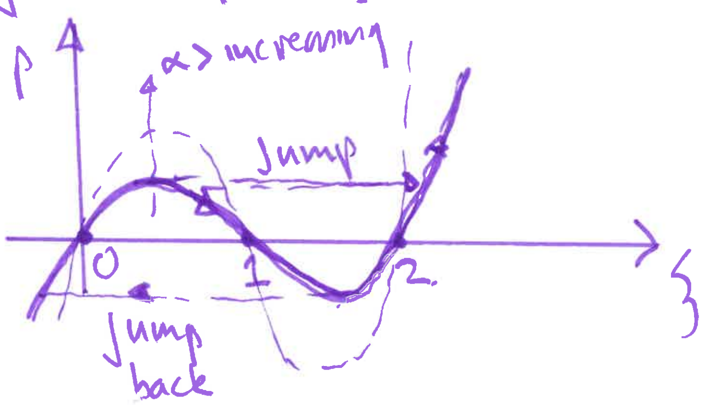
Given the form in (a.iii), it's worth checking when $\xi = 0, 1, 2$:

$\xi = 0$: $P = 4kL\left[\sin\alpha - \cos\alpha\tan\alpha\right] = 4kL\left[\sin\alpha - \sin\alpha\right] = 0$

$\xi = 1$: $P = 4kL\left[0 - \cos\alpha \cdot 0\right] = 0$

$\xi = 2$: $P = 4kL\left[-\sin\alpha - \cos\alpha \cdot -\tan\alpha\right] = 4kL\left[-\sin\alpha + \sin\alpha\right] = 0$

Thus, we operate somewhere from $\xi = 0$ to $\xi > 2$: try plotting some points.



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2 (a.iii) for α small: then $\sin(1-\xi)\alpha$, $\tan(1-\xi)\alpha$, $\cos\alpha$ all small

$\sin X \simeq X - X^3/6$; $\cos\alpha = 1-\alpha^2/2$; $\tan X = X + X^3/3$

$X = (1-\xi)\alpha$

$$\bar{P} = P/4kL \simeq \sin X - \cos\alpha . \tan X = X - X^3/6 - (1-\alpha^2/2)(X + X^3/3)$$

$$\Rightarrow \bar{P} = \cancel{X} - X^3/6 - \cancel{X} - X^3/6 + \alpha^2/2\,(X + X^3/3)$$
$$= -X^3/2 + \alpha^2/2\,X + \cancel{\alpha^2 X^3/6} \text{ small (higher order)}$$

$$\bar{P} = X/2\,[\alpha^2 - X^2] = \frac{(1-\xi)\alpha}{2}\left[\alpha^2 - (1-\xi)^2\alpha^2\right]$$

$$\bar{P} = (1-\xi)\,\alpha^3/2 \cdot [1-(1-\xi)^2] ; \quad 1-(1-\xi)^2 = 1-(1-2\xi+\xi^2) = 2\xi - \xi^2$$

$$\Rightarrow P = 4kL \cdot (1-\xi)\,\alpha^3/2 \cdot \xi(2-\xi)$$

$$\Rightarrow \underline{P = 2kL\alpha^3 . \xi(1-\xi)(2-\xi)}$$

2 b) Potential energy becomes $U = \tfrac{1}{2}ke^2 - Pv + \tfrac{1}{2}c\theta^2$ relative rotation

$\Rightarrow \theta = 2\alpha - 2(1-\xi)\alpha = -2\xi\alpha$ -ve because relative inclination decreases.

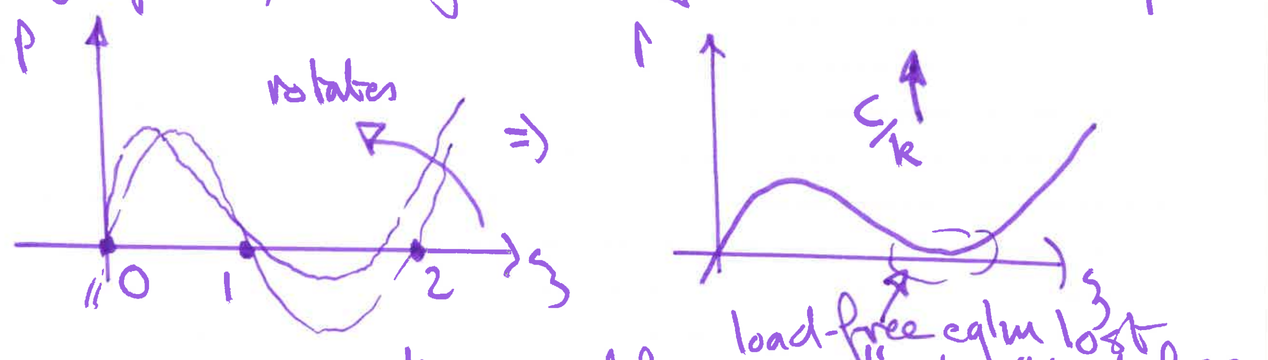
$\Rightarrow U = \ldots + \tfrac{1}{2}c(4\xi\alpha)^2 = 2c\xi^2\alpha^2$

$\partial U/\partial\xi = \ldots 4c\alpha^2.\xi$; a linear term, which adds to P

$$\Rightarrow PL\cos(1-\xi)\alpha - 4kL^2\sin(1-\xi)\alpha\,[\cos(1-\xi)\alpha - \cos\alpha] \underbrace{- 4c\alpha^2\xi}_{\text{new}} = 0$$

$$\Rightarrow P = \underbrace{4kL[\ldots]}_{\text{before}} + \underbrace{\frac{4c\alpha}{L}\cdot\frac{\xi}{\cos(1-\xi)\alpha}}_{\text{new}} \simeq \frac{4c\alpha}{L}.\xi \text{ if } \alpha \text{ small (linear)}$$

The extra term adds a linear term to the previous P-ξ graph, thereby "rotating" variation about origin



$\Rightarrow$ the moment from spring "undoes" load-free states.

Q3)

a) Moment M_A when rotation prevented, from Databook

$$M_A = \frac{W\,2/27\,L^3}{L^2} + \frac{W\,4/27\,L^3}{L^2} = \frac{2}{9}WL$$

Stiffness of joint A, from Data sheet

$$K_{AA} = 8\frac{EI}{L} + 4\frac{EI}{L} = 12\frac{EI}{L}$$

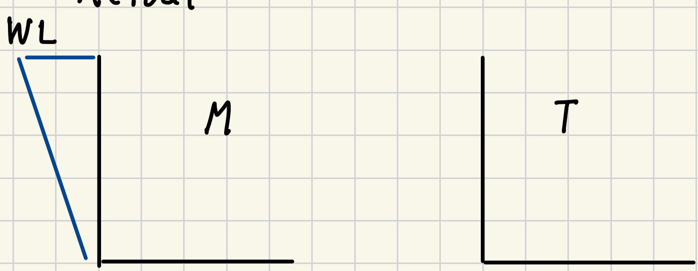
$$\Rightarrow \quad \beta = \frac{6\,WL}{27}\cdot\frac{L}{12EI} = \frac{1\,WL^2}{54EI}$$

b)

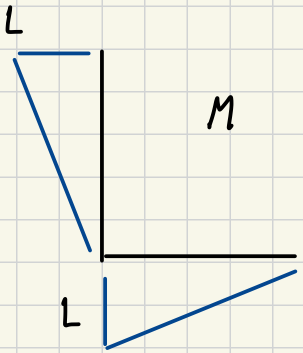
i) from Databook $\quad \delta_B = \dfrac{WL^3}{3EI}$

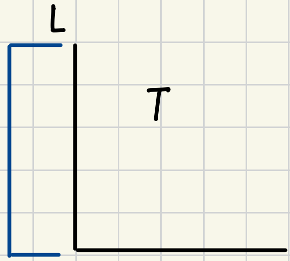
ii) Principle of virtual disp.

Actual



Virtual





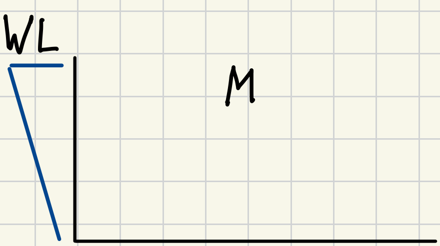
$$\delta_B^a = \frac{1}{EI} \frac{WL^3}{3}$$

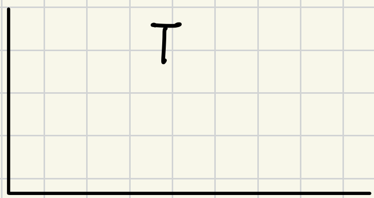
$$\delta_B^v = \frac{2}{EI} \frac{L^3}{3} + \frac{1}{GJ} L^3 = \frac{7}{6EI} L^3$$

Reaction force: $X = \dfrac{\delta_{BA}}{\delta_{BV}} = \dfrac{2}{7} W$

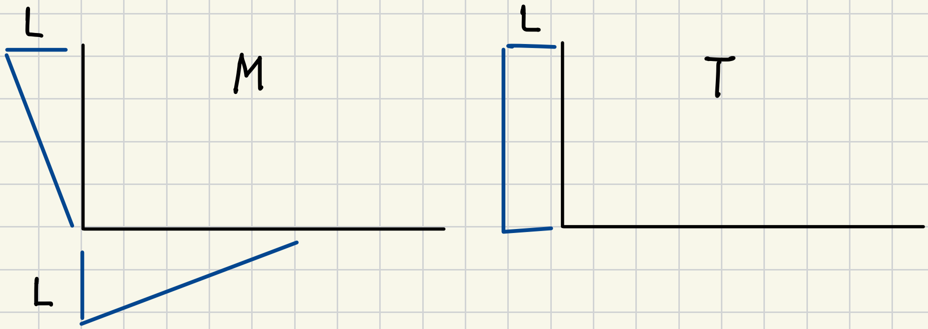
c) Degree of statical indeterminacy is 2

Actual

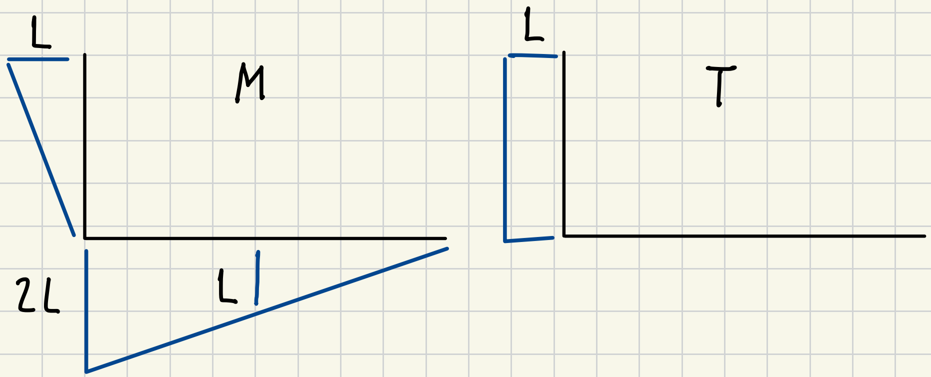




Virtual ("1" at B)



Virtual ("1" at C)



$$EI\,\delta^{v}_{BB} = 2\,\frac{L^3}{3} + \frac{L^3}{2} = \frac{7}{6}L^3$$

$$EI\,\delta^{v}_{BC} = \frac{L^3}{3} + \frac{2L^3}{3} + \frac{L^3}{6} + \frac{L^3}{2} = \frac{5}{3}L^3$$

$$EI\,\delta^{v}_{CC} = \frac{L^3}{3} + \frac{8L^3}{3} + \frac{L^3}{2} = \frac{7}{2}L^3$$

$$EI\,\delta_B^a = \frac{WL^3}{3}$$

$$EI\,\delta_c^a = \frac{WL^3}{3}$$

$$\begin{pmatrix} 7/6 & 5/3 \\ 5/3 & 7/2 \end{pmatrix} \begin{pmatrix} X_b \\ X_c \end{pmatrix} = \begin{pmatrix} W/3 \\ W/3 \end{pmatrix}$$

$$\Rightarrow X_b = \frac{22W}{47} \qquad X_a = -\frac{6W}{47}$$

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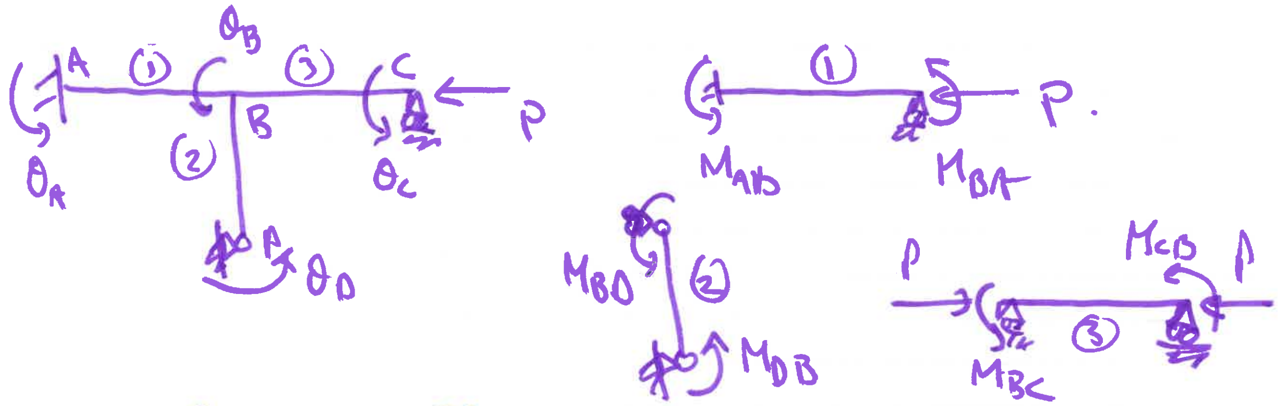
4(a)



①: $\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = k\begin{bmatrix} s & sc \\ sc & s \end{bmatrix}\begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$; $\theta_A = 0 \Rightarrow$ $\underline{M_{BA} = ks\theta_B}$ —①.

② $\begin{bmatrix} M_{BD} \\ M_{DB} \end{bmatrix} = k\underbrace{\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}}_{\text{no axial force}}\begin{bmatrix} \theta_B \\ \theta_D \end{bmatrix} \Rightarrow M_{DB} = 0$ (pinned)

$\underline{\theta_D = -\theta_B/2}$ —②

$\Rightarrow M_{BD} = k[4\theta_B + 2\theta_D] = \underline{3k\theta_B}$

③ $\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = k\begin{bmatrix} s & sc \\ sc & s \end{bmatrix}\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$ to be determined:

Eqlm of joint B: M_{BA} M_{BC} M_{BD} $\Rightarrow M_B = M_{BA} + M_{BC} + M_{BD}$.

$\Rightarrow \begin{bmatrix} M_B \\ M_C \end{bmatrix} = \begin{bmatrix} M_{BA} + M_{BC} + M_{BD} \\ M_{CB} \end{bmatrix} = k\begin{bmatrix} s+s+3 & sc \\ sc & s \end{bmatrix}\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$

$\Rightarrow \underline{\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \begin{bmatrix} 2s+3 & sc \\ sc & s \end{bmatrix}\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}}$.

Buckling occurs when stiffness matrix determinant $= 0$

$\Rightarrow (2s+3)s - sc \times sc = 0 \quad 2s^2 + 3s - s^2c^2 = 0$

$\Rightarrow \underline{s = 0}$ or $\underline{2s+3-sc^2} = 0$.

$s = 0 \Rightarrow P/P_E \approx 2.0$ (just above 2) : check $2s+3-sc^2$ for values of P/P_E lower than 2.

$P/P_E = 1$; $s = 2.47$, $c = 1 \Rightarrow 2s+3-sc^2 = 6.93$

$P/P_E = 1.4$; $s = 1.68$, $c = 1.66 \Rightarrow$ " $= 3.6$

$P/P_E = 1.6$; $s = 1.224$, $c = 2.44$ " $= -0.4812$

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Select / try $P/P_E = 1.55$: try interpolating between tabulated values.

$s \simeq 1.335$; $c \simeq 2.173 \Rightarrow 2s - c^2 + 3 \simeq 0.948$

True value of P/P_E between 1.55 and 1.6: can iterate further but this (1.55) is reasonable.

$$\Rightarrow P = \underline{1.55 P_E} = 1.55\pi^2 EI/L^2 = \underline{\underline{15.3\frac{EI}{L^2}}}$$

$(< 2P_E \checkmark)$

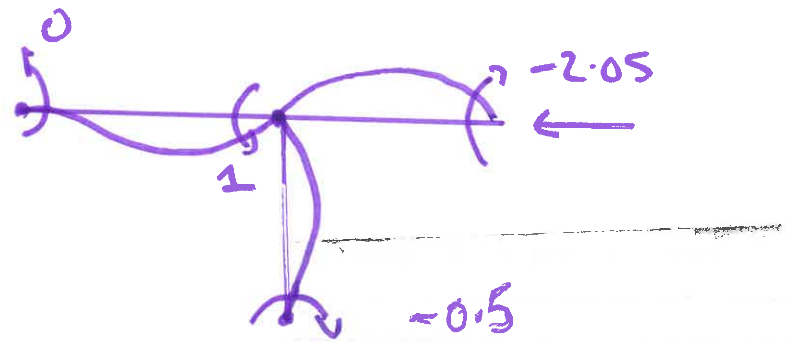
4(b) $2s+3 = 5.67$ and $sc = 2.9$ using $s = 1.335$, $c = 2.173$

$$\Rightarrow \text{Stiffness matrix: } \begin{bmatrix} 5.67 & 2.9 \\ 2.9 & 1.335 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = 0$$

$\Rightarrow 5.67\theta_B + 2.9\theta_C = 0$ or $2.9\theta_B + 1.335\theta_3 = 0$

$\Rightarrow \theta_C = -1.96\theta_B$ or $\theta_C = -2.17\theta_B$

Take average $\Rightarrow \underline{\underline{\theta_C \simeq -2.05\theta_B}}$ c.f. $\underline{\underline{\theta_A = -\theta_B/2}}$



4(c) If pin at B then BC is simply pinned at each end and buckles locally according to

$$\underline{\underline{P = P_E}} = \pi^2 EI/L^2.$$

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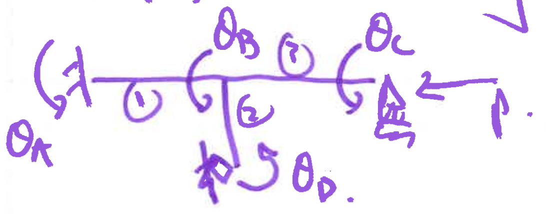
4 (a) For (a) alternatively, treat B as the sole exposed joint.



(1) $\Rightarrow$ $M_{BA} = ks\theta_B$ as before

(2) $\Rightarrow$ $M_{BD} = 3k\theta_B$ and $\theta_D = -\frac{\theta_B}{2}$ as before.

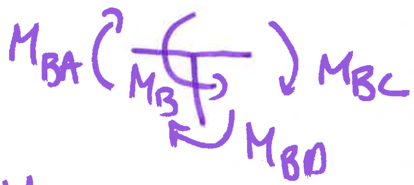
(3)

$$\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = k \begin{bmatrix} s & sc \\ sc & s \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

$\Rightarrow$ $M_{CB} = 0$, pinned $\Rightarrow$ $\theta_C = -c\theta_B$

$\Rightarrow$ $M_{BC} = k[s\theta_B + sc.(-c\theta_B)] = \underline{\underline{ks\theta_B(1-c^2)}}$ (3)

Eqlm of B joint



$M_B = M_{BA} + M_{BC} + M_{BD} = 0$

$\Rightarrow$ $k\theta_B s + k\theta_B s(1-c^2) + 3k\theta_B = 0$

$k\theta_B[s + s - sc^2 + 3] = 0$

$\Rightarrow$ $\underline{\underline{2s + 3 - sc^2 = 0}}$ as before