Question 1

1. (a) Weak form admits less regular (reduced continuity) functions. This is important for FE methods as it permits use of piecewise polynomials (C^0 continuous)

(b) (i) Apply integration by parts,

$$\int -wu'' + wku - wf \, dx + wu'\Big|_0^L = 0$$

$$\int w(-u'' + ku - f)\, dx = 0$$

Since it must hold for all w,

$$u'' - ku = f \qquad (\alpha)$$

Since $w(0) = w(L) = 0$, then must be Dirichlet condition at $x=0$ and $x=L$.

(ii) $$u = c_1 \exp(\sqrt{k}\, x) + c_2 \exp(-\sqrt{k}\, x)$$
$$u'' = c_1 k \exp(\sqrt{k}\, x) + c_2 k \exp(-\sqrt{k}\, x)$$

This solves (α) when $f=0$.

For boundary conditions,

$$u(0) = c_1 + c_2$$
$$u(L) = c_1 k \exp(\sqrt{k} L) + c_2 k \exp(-\sqrt{k} L)$$

(c) (i) If $e_0 = 0$ (L^2), then $e_1 = 0$ (H^1-norm) also. To capture exact solution, FE space (basis functions) must contain the exact solution

(ii) For linear element, convergence rate is $O(h^2)$, therefore expected reduction in error is $4^2 = 16 \Rightarrow e_0/16$

For H^1-norm, $O(h) \Rightarrow e_1/4$

Comment.

Q1 Weak and strong forms and Finite element method in 1D
47 attempts, Average mark 12.6/20, Maximum 18, Minimum 2.

A popular and straightforward question, well-answered by candidates. The biggest difficulty for students is that some of them were not able to deduce the strong form from the weak one.

Question 2

2. (a) Implicit. Heat equation is a 'stiff' equation, which means the time step restriction is severe, i.e. $O(h^2)$. In practice, this means the stash time step is prohibitively small.

(b) (i) With appropriate normalisation,

$$\underline{u}_i^T \underline{\underline{K}}\, \underline{u}_j = \begin{cases} \lambda_j & i=j \\ 0 & i\neq 0 \end{cases}$$

$$\underline{u}_i^T \underline{\underline{M}}\, \underline{u}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

Inserting $\underline{x}(t) = \sum_i \alpha_i \underline{u}_i$ into (2),

$$\underline{\underline{M}} \sum_{i=1}^{n} \dot{\alpha}_i \underline{u}_i + \underline{\underline{K}} \sum_{i=1}^{n} \alpha_i \underline{u}_i = \underline{0}$$

Pre-multiply by $\underline{u}_j^T$,

$$\underline{u}_j^T \underline{\underline{M}} \sum_{i=1}^{n} \dot{\alpha}_i \underline{u}_i + \underline{u}_j^T \underline{\underline{K}} \sum_{i=1}^{n} \alpha_i \underline{u}_i = \underline{0}$$

By orthogonality,

$$\dot{\alpha}_i + \lambda_i \alpha_i = 0 \quad \text{for } i = 1, \dots, n \qquad (\beta)$$

(ii) Require initial conditions $(\underline{x}(0)) \Rightarrow \alpha_i(0)$ to eliminate constant from (β).

(iii) Forward Euler: $\quad y_{n+1} = y_n + \Delta t \dot{y}_n$

Applying to (β):

$$\alpha_{n+1} = \alpha_n - \Delta t \lambda \alpha_n$$

$$\alpha_{n+1} = \underbrace{(1 - \Delta t \lambda)}_{A:\ \text{amplification factor}} \alpha_n$$

Need $|A| \leq 1$. Note that λ & Δt are positive.

$$(1 - \Delta t \lambda) = -1 \Rightarrow \Delta t = 2/\lambda$$

$$\Delta t_{crit} = 2/\lambda_{max}$$

(iv) Solving the generalised eigenvalue problem

(v) Problem would have form $\underline{\underline{M}}\dot{\underline{x}} + \underline{F}(\underline{x}) = \underline{0} \Rightarrow$ not a linear system.
No relevant eigenvalue problem to solve.

Comment :

Q2 Solution of a linear heat equation by a modal analysis

36 attempts, Average mark 12.0/20, Maximum 18, Minimum 5.

An unpopular question (an evidence that they did not know the material in the lecturenotes well). Few understood the significance of the characteristics of the generalised eigenvalue problem. Many students resorted to the reverse engineering approach to answer the question by jumping steps.

Question 3

(a) The force balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0 \qquad (1)$$

BCs

on the left, $\underline{u} = (0,0)$,

On the right, $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}$,

on the top & bottom, $\underline{\underline{\sigma}} \cdot \underline{n} = 0$.

Multiply (1) by w_x and integrate

$$\int_\Omega \underbrace{w_x \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right)}_{\frac{\partial}{\partial x}(w_x \sigma_{xx}) - \sigma_{xx}\frac{\partial w_x}{\partial x} + \frac{\partial}{\partial y}(w_x \sigma_{xy}) - \sigma_{xy}\frac{\partial w_y}{\partial y}} + w_x f_x \, d\Omega = 0$$

Apply the divergence theorem

$$\int_\Gamma w_x \underbrace{(\sigma_{xx} n_x + \sigma_{xy} n_y)}_{t_x} d\Gamma + \int_\Omega w_x f_x - \sigma_{xx}\frac{\partial w_x}{\partial x} - \sigma_{xy}\frac{\partial w_x}{\partial y} \, d\Omega = 0$$

or $$\int_\Omega \frac{\partial w_x}{\partial x}\sigma_{xx} + \frac{\partial w_x}{\partial y}\sigma_{xy} \, d\Omega = \int_\Omega w_x f_x \, d\Omega + \int_\Gamma w_x t_x \, d\Gamma$$

Similarly, we can multiply the equation in the y-direction with w_y and obtain

$$\int_\Omega \frac{\partial w_y}{\partial x}\sigma_{xy} + \frac{\partial w_y}{\partial y}\sigma_{yy} = \int_\Omega w_y f_y \, d\Omega + \int_\Gamma w_y t_y \, d\Gamma.$$

(3-2)

(b)

(i) $\underline{f} = \begin{pmatrix}0\\-10\end{pmatrix}$, $\quad \underline{f}_3 = \int_\Omega N_3 \underline{f}\, d\Omega = \begin{pmatrix}0\\ -10\int_\Omega N_3\, d\Omega\end{pmatrix}$

use one Gauss quadrature point.

$$\int_\Omega N_3\, d\Omega = \int_{-1}^{1}\int_{-1}^{1} N_3(0,0)\,|J|\, d\xi\, d\eta$$

$$\approx 4\, N_3(0,0)\, |J(0,0)|$$

determinant of Jacobian

$$J^e(0,0) = \begin{bmatrix} (\cancel{\eta}-1)/4 & (1-\cancel{\eta})/4 & (1+\cancel{\eta})/4 & (-1-\cancel{\eta})/4 \\ (\cancel{\xi}-1)/4 & (-1-\cancel{\xi})/4 & (1+\cancel{\xi})/4 & (1-\cancel{\xi})/4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & \frac{1}{2} \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -1/4 & 1/4 & 1/4 & -1/4 \\ -1/4 & -1 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & \frac{1}{2} \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{8} \\ 0 & \frac{3}{8} \end{bmatrix}, \quad |J^e(0,0)| = \frac{3}{8}$$

$N_3(0,0) = \frac{1}{4}$,

$$-10\int_\Omega N_3\, d\Omega \approx -10 \times 4 \times \frac{1}{4} \times \frac{3}{8} = \boxed{\frac{-15}{4}}$$

(ii) $\underline{t} = \begin{pmatrix}0.5\\ y\end{pmatrix}$ $\quad \underline{t}_3 = \int_\Gamma N_3 \begin{pmatrix}0.5\\ y\end{pmatrix} d\Gamma = \begin{pmatrix}\int_\Gamma N_3 \times 0.5\, d\Gamma \\ \int_\Gamma N_3\, y\, d\Gamma\end{pmatrix}$

The only boundary for the integrale is not zero is the right boundary, on which, $y \in [\frac{1}{2}, 1]$

$y = \frac{1}{2}s + \frac{1}{2}$, $\quad \int_\Gamma N_3\, y\, d\Gamma = \int_0^1 s\left(\frac{1}{2}s + \frac{1}{2}\right)\frac{1}{2}\, ds$

$dy = \frac{1}{2} ds$

$$= \frac{1}{4}\int_0^1 s^2 + s\, ds = \frac{1}{4}\left(\frac{s^3}{3} + \frac{s^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{4}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{4} \times \frac{5}{6} = \boxed{\frac{5}{24}}$$

$$\int_\Gamma N_3 \times 0.5\, dy = 0.5\int_0^1 s \times \frac{1}{2}\, ds = \frac{1}{4}\left.\frac{s^2}{2}\right|_0^1$$

$$= \frac{1}{4} \times \frac{1}{2} = \boxed{\frac{1}{8}}$$

Comment

Q3 Calculation of stress in an elasticity problem

31 attempts, Average mark 11.9/20, Maximum 19, Minimum 2.

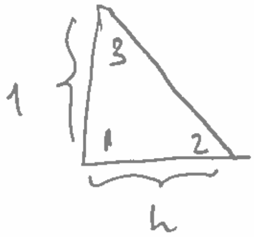
The least popular question that involves long calculations. Few students could use the divergence (Gauss) theorem correctly in order to deduce the weak form from the strong one, even though this is a part from the lecturenotes. Not knowing the material in the lecturenotes well is a common characteristic among our students.

Question 4

4-1

(a) Element [1]

(i) The area $= \frac{h^2}{2}$



$$N_1 = 1 - \frac{x}{h} - \frac{y}{h}$$

$$N_2 = \frac{x}{h} \quad , N_3 = \frac{y}{h}$$

$$\underline{\underline{B}} = \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} & 0 \\ -\frac{1}{h} & 0 & \frac{1}{h} \end{bmatrix}$$

(ii) $$\underline{\underline{B}}^T \cdot \underline{\underline{B}} = \begin{bmatrix} -\frac{1}{h} & -\frac{1}{h} \\ \frac{1}{h} & 0 \\ 0 & \frac{1}{h} \end{bmatrix} \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} & 0 \\ -\frac{1}{h} & 0 & \frac{1}{h} \end{bmatrix}$$

$$= \frac{1}{h^2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{K}} = \int_{[1]} \underline{\underline{B}}^T \cdot \underline{\underline{B}} \, dxdy = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(iii) $$f_2 = \int_0^h \left(\int_0^{h-x} \frac{x}{h} \, dy \right) dx = \frac{1}{h} \int_0^h xh - x^2 \, dx$$

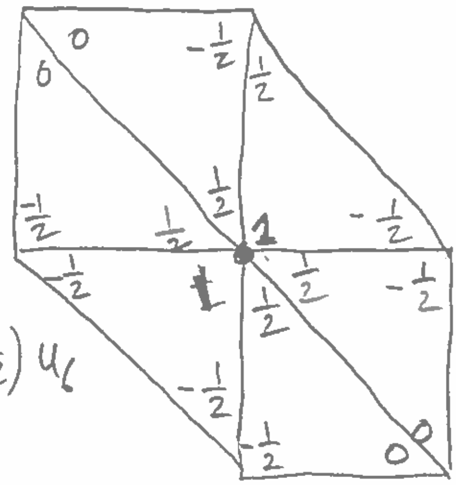
$$= \frac{1}{h} \left(\frac{hx^2}{2} - \frac{x^3}{3} \right) \Big|_0^h = \frac{h^2}{6}$$

$f_3 = f_2$ by symmetry

$$f_1 = \int_{[1]} 1 - \frac{x}{h} - \frac{y}{h} \, dxdy = \int_{[1]} 1 \, dxdy - f_2 - f_3$$

$$= \frac{h^2}{2} - \frac{h^2}{6} - \frac{h^2}{6} = \frac{h^2}{6}$$

(c) Hence the coefficients of the finite element discretisation related to node 1 is shown in the figure $\Longrightarrow$



$$\left(1+\tfrac{1}{2}+\tfrac{1}{2}+1+\tfrac{1}{2}+\tfrac{1}{2}\right)u_1-\left(\tfrac{1}{2}+\tfrac{1}{2}\right)u_2$$
$$-\left(\tfrac{1}{2}+\tfrac{1}{2}\right)u_3-\left(\tfrac{1}{2}+\tfrac{1}{2}\right)u_5-\left(\tfrac{1}{2}+\tfrac{1}{2}\right)u_6$$
$$=\tfrac{1}{6}h^2\times 6=h^2$$

$$\Rightarrow \quad 4u_1-u_2-u_3-u_5-u_6=h^2$$

$$u_1=\tfrac{1}{4}\left(h^2+\overset{=1}{u_2}+\overset{=0}{u_3}+\overset{=0}{u_5}+\overset{=0}{u_6}\right)$$
$$=\boxed{\tfrac{1}{4}\left(h^2+1\right)}$$

Comment:

Q4 Finite element solution of a heat equation problem
64 attempts, Average mark 12.6/20, Maximum 20, Minimum 6.

A popular question involving the full procedure of the finite element method, well-answered by candidates. The only difficulty was in the assembly of the finite element matrix, on which most students did surprisingly well. This means that they have understood the essence of the finite element method.