EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 25 April 2018 2 to 3.40

Module 3D7

FINITE ELEMENT METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 Data Sheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) A key step in finite element methods is considering the *weak* form of a differential equation. Remark on key differences between the *strong* and *weak* forms of an equation in the context of finite element methods. [20%]

(b) The weak form of a particular problem on the interval $(0, L)$ reads: find u such that

$$\int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + wku \, \mathrm{d}x = \int_0^L wf \, \mathrm{d}x \tag{1}$$

for all admissible w, where $k \geq 0$ is a constant and f is prescribed. It is required that $w = 0$ at both $x = 0$ and $x = L$.

(i) Find the strong form corresponding to Eqn. (1), and any boundary conditions implied by Eqn. (1). [30%]

(ii) Show that $u = c_1 \exp(\sqrt{k}x) + c_2 \exp(-\sqrt{k}x)$, where c_1 and c_2 are constants, is a solution to Eqn. (1) when $k > 0$ and $f = 0$, and find expressions for what the boundary conditions must be. [20%]

(c) A new implementation of the finite element method is used to compute approximate solutions to Eqn. (1). To test the implementation, problems with known analytical solutions are considered and the error computed in the L^2- and H^1-norms.

(i) It is observed for one case that the error in the L^2-norm is zero. For this case, what can you say about the error in the H^1-norm, and about the finite element shape functions with respect to the exact solution? [15%]

(ii) If the error on a given mesh using linear elements is e_0 in the L^2-norm and e_1 in the H^1-norm, what errors would you predict after refining the mesh such that the element size is reduced by a factor of four? [15%]

2 For a particular unsteady linear heat conduction problem, the finite element problem in matrix form is

$$\boldsymbol{M}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{M}$ and $\boldsymbol{K}$ are $n \times n$ matrices and $\boldsymbol{x}$ is the vector of temperature degrees-of-freedom.

(a) Would you recommend an explicit or implicit time integration method for this problem? Justify your answer. [10%]

(b) The problem in Eqn. (2) is to be solved using a modal analysis approach. The eigenvectors $\boldsymbol{u}_i$ and eigenvalues λ_i satisfy the generalised eigenvalue problem

$$(\boldsymbol{K} - \lambda_i \boldsymbol{M})\boldsymbol{u}_i = \boldsymbol{0},$$

and the finite element solution is expressed as $\boldsymbol{x}(t) = \sum_{i=1}^{n} \alpha_i(t)\boldsymbol{u}_i$.

(i) Show that α_i is found by solving

$$\dot{\alpha}_i + \lambda_i \alpha_i = 0.$$

[40%]

(ii) To completely define the response $\boldsymbol{x}(t)$, what extra information is required? [10%]

(iii) Determine the critical time step in terms of λ_i for the forward Euler method. [20%]

(iv) What is the dominant computational cost in solving this problem using modal analysis? [10%]

(v) If the thermal conductivity was to depend on temperature, explain why modal analysis could not be used. [10%]

3 Consider the linear elastic cantilever shown in Fig. 1. The cantilever is subject to a body force $\boldsymbol{f} = (f_x, f_y)^T$. The left-hand boundary is fixed with $\boldsymbol{u} = (0,0)^T$. The top and bottom boundaries are traction free. On the right-hand boundary a traction $\boldsymbol{t} = (t_x, t_y)^T$ is prescribed.

(a) Let $\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})^T$ be the stress vector. The equilibrium equations of the cantilever in component form are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0 \,,$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0 \,.$$

Deduce the weak formulation of the equilibrium equations in component form. [30%]

(b) The cantilever is discretised with one four-noded quadrilateral element, as shown in Fig. 1.

(i) For $\boldsymbol{f} = (0, -10)^T$, calculate numerically the external force due to $\boldsymbol{f}$ associated with node 3 using one Gauss point. [40%]

(ii) For an applied traction $\boldsymbol{t} = (0.5, y)^T$, calculate the external force due to $\boldsymbol{t}$ associated with node 3. [30%]

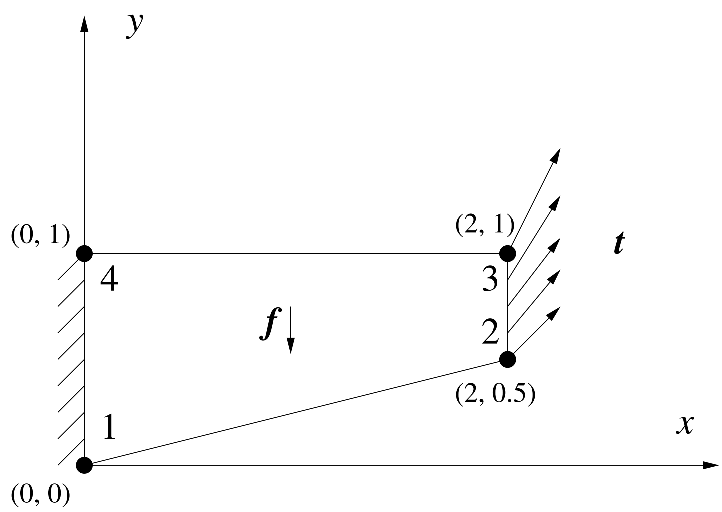

Fig. 1

4 Consider the heat equation

$$\nabla^2 T = -1 \tag{3}$$

in the hexagonal domain in Fig. 2, where $h > 0$ is a constant. A Dirichlet boundary condition is prescribed on the boundary of the hexagon. We seek the finite element solution of the heat equation using three-noded triangular elements, with the hexagon divided into 6 triangles; the numbering of the elements and vertices is shown in Fig. 2.

(a) Let $\boldsymbol{N}^e$ be the shape functions of the three-noded triangular element [1] in Fig. 2. For this element:

(i) Calculate the $\boldsymbol{B}^e$ matrix. [20%]

(ii) Calculate the element conductance matrix

$$\boldsymbol{K}^e = \int_{[1]} \boldsymbol{B}^{eT} \boldsymbol{B}^e \, d\Omega \, .$$

[20%]

(iii) Calculate the element source flux vector

$$\boldsymbol{f}^e = \int_{[1]} \boldsymbol{N}^{eT} \, d\Omega \, .$$

[20%]

(b) Let the temperature T be equal to 1 at node 2 ($T_2 = 1$) and 0 at all other boundary nodes ($T_i = 0$, $i = 3, ..., 7$). Find the temperature value T_1 at node 1 in the finite element solution of Eqn. (3). [40%]

Hint: Note that element conductance matrices and source fluxes do not change under translation and rotation of an element. You may use this fact to simplify your calculations.

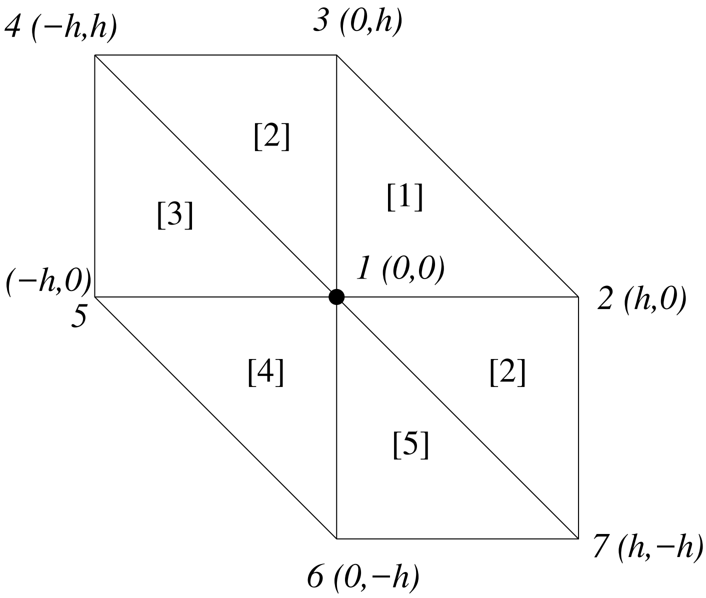

Fig. 2

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Engineering Tripos Part IIA
Module 3D7: Finite Element Methods

Data Sheet

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$
Element force vector	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\, dV + \int_{\Gamma_e} \boldsymbol{N}^T \boldsymbol{t}\, d\Gamma$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI \boldsymbol{B}\, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

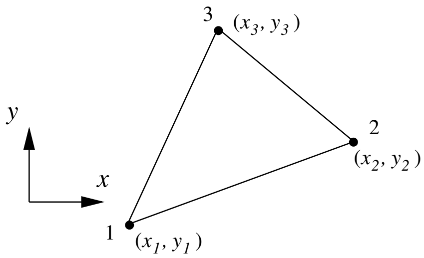
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D)

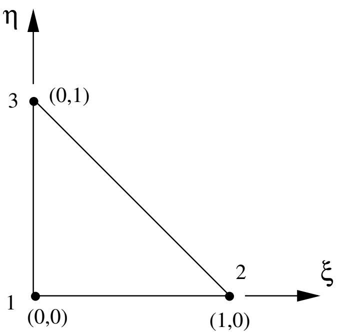
$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Shape functions

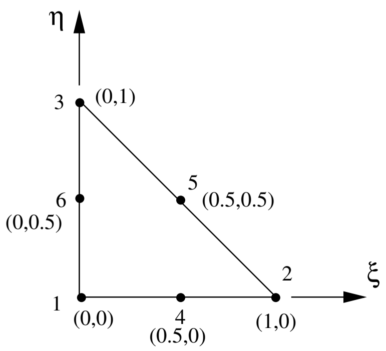


$$N_1 = \big((x_2 y_3 - x_3 y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y\big)/2A$$
$$N_2 = \big((x_3 y_1 - x_1 y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y\big)/2A$$
$$N_3 = \big((x_1 y_2 - x_2 y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y\big)/2A$$

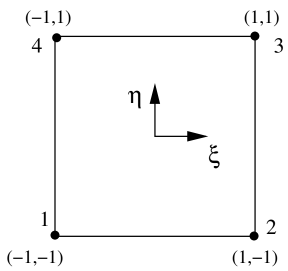
$A =$ area of triangle



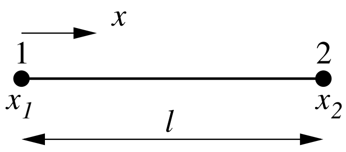
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\big(1 - \xi - \eta\big)^2 - \big(1 - \xi - \eta\big)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\big(1 - \xi - \eta\big)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\big(1 - \xi - \eta\big)$$



$$N_1 = (1 - \xi)\big(1 - \eta\big)/4$$
$$N_2 = (1 + \xi)\big(1 - \eta\big)/4$$
$$N_3 = (1 + \xi)\big(1 + \eta\big)/4$$
$$N_4 = (1 - \xi)\big(1 + \eta\big)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1,1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$