Question 1

(a) Note the foce at $x = 0$ is zero and the force at $x = L$ is $-ku(x = L)$. Therefore, Boundary conditions:

$$EA\frac{\partial u}{\partial x}(0,t) = 0,$$
$$EA\frac{\partial u}{\partial x}(L,t) = -ku(L,t).$$

At $t = 0$, the bar at time zero travels at velocity v_0, therefore the initial conditions are:

$$u(x,0) = 0,$$
$$\dot{u}(x,0) = v_0.$$

(b) The governing equation of the bar is given by

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A\ddot{u}$$

Therefore, let $w = w(x)$ be the weight function, the weak formualtion is given by

$$\int_0^L EA\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}dx + \int_0^L \rho A\ddot{u}w dx = wEA\frac{\partial u}{\partial x}\Big|_0^L.$$

Plug in the Neumann boundary condition from part (a), the weak formulatoin of the problem is given by

$$\int_0^L EA\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}dx + \int_0^L \rho A\ddot{u}w dx = -wku(L,t).$$

(c) The stiffness matrix of a single element is given by

$$K = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

while the element mass matrix is given by

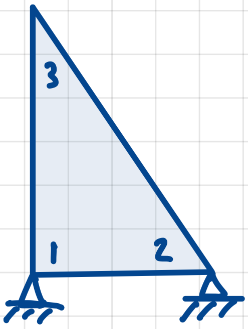
$$M = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Let u_1, u_2 denote the nodal displacement of the element at $x = 0$ and $x = L$, respectively. The semi discrete formulation of the problem is then

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -ku_2 \end{bmatrix}.$$

Q2a) 2)

i)



$$N_3(y) = \frac{1}{3}y$$

$$B^e = \begin{pmatrix} \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$$

ii)

$$D = \frac{200}{1.3 \cdot 0.4}\begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} = 384.6\begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}$$

$$K = 3 \cdot 384.6 \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$$

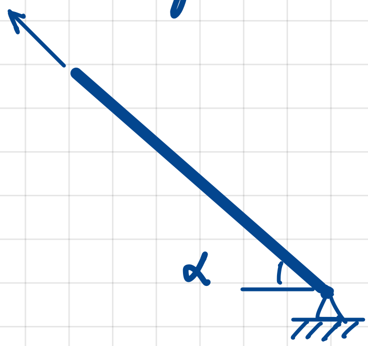
$$= \begin{pmatrix} 25.64 & 0 \\ 0 & 89.74 \end{pmatrix}$$

iii) Area increases by 1.5^2; shapefunction derivs decrease by 1.5 so that

$$= \begin{pmatrix} 25.64 & 0 \\ 0 & 89.74 \end{pmatrix}$$

Q2b)

i) Length $= 3.75$



$$K = \frac{100}{3.75}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 26.67 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Rotation matrix} = \begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$$

$$\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}\begin{pmatrix} 26.67 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix} = \begin{pmatrix} 17.06 & -12.80 \\ -12.80 & 9.60 \end{pmatrix}$$

ii) $$\begin{pmatrix} 25.64+17.06 & -12.80 \\ -12.80 & 89.74+9.60 \end{pmatrix} = \begin{pmatrix} 42.71 & -12.80 \\ -12.80 & 80.14 \end{pmatrix}$$

$$\begin{pmatrix} 42.71 & -12.80 \\ -12.80 & 80.14 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} -f_x \\ 0 \end{pmatrix}$$

$$\Rightarrow u_x = -0.0223 f_x$$
$$u_y = 0.00357 f_x$$

Q3a)

i) $N_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \\ 7 \\ 4 \\ 0 \\ 4 \end{pmatrix}$$

$x = N_1 + 5N_2 + 7N_3$

$y = 4N_3 + 4N_4$

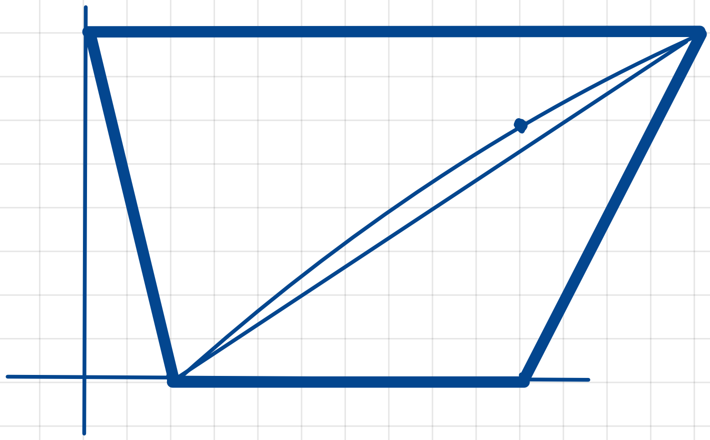
$\Longrightarrow \quad x = \frac{1}{4}(11\xi + \eta + 3\xi\eta + 13)$

$y = 2(1+\eta)$

ii) $x = \frac{1}{4}\left(\frac{11}{2} + \frac{1}{2} + \frac{3}{4} + 13\right) = \frac{79}{16} \approx 4.94$

$y = 2\left(1 + \frac{1}{2}\right) = 3$

iii)



Line is mapped to a quadratic function

iv) The mapping from the physical element to the parent element requires the solution of a nonlinear equation. That is we must compute the roots of a nonlinear function, which can be done by e.g. plotting and trial and error.

Q3b)

In index notation:

$$\int_\Omega u_{,i} w_{,i} + \int_\Omega c_i u_{,i} w + \int_\Omega f w = 0$$

$$\int_\Omega (u_{,i} w)_{,i} = \int_\Omega u_{,ii} w + \int_\Omega u_{,i} w_{,i}$$

$$-\int_\Omega u_{,ii} w + \int_\Gamma w u_{,i} n_i + \int_\Omega c_i u_{,i} w + \int f w = 0$$

$$\int (-u_{,ii} + c_i u_{,i} + f) w + \int_\Gamma w u_{,i} n_i = 0$$

$$-u_{,ii} + c_i u_{,i} + f = 0 \quad \text{on } \Omega$$

$$u_{,i} n_i = 0 \quad \text{on } \Gamma_N$$

$$u = 0 \quad \text{on } \Gamma_D$$

Question 4

(a) A priori error estimation is the analysis of the errors in a finite element simulation before actually performing an analysis. It can tell whether the finite element solution will converge to the exact solution as the mesh is refined, and how quickly it will converge if the element type is changed or the mesh is refined.
A posterior error estimation involves quantifying the error after a simulation has been performed. With an estimate of the error, a judgement can be made on the reliability of the result.
(b) The two strategies are the h-adaptivity and p-adaptivity. H-adaptivity changes the finite element mesh, while p-adaptivity changes the polynomial order of the finite element shape functions to reduce the error.
(c) The error estimate is given by

$$\|u - u_h\|_s \leq Ch^{\beta}\|u\|_{k+1}.$$

Therefore, with polynomial order $k = 3$, L_2-norm $s = 2$, $\beta = k + 1 - s = 2$, the a-priori error estimate is

$$\|u - u_h\|_2 \leq Ch^2\|u\|_4.$$

(d) The Crank-Nicolson approximation of $\dot{y}$ is given by

$$y_{n+1} = y_n - 0.5\lambda\Delta t(y_n + y_{n+1}),$$

which can be rewritten as

$$y_{n+1} = \frac{1 - 0.5\lambda\Delta t}{1 + 0.5\lambda\Delta t} y_n \, .$$

Hence, the amplitude factor is given by

$$A = \left| \frac{1 - 0.5\lambda\Delta t}{1 + 0.5\lambda\Delta t} \right| < 1.$$

Consequently, the solution is unconditionally stable.