Version FC/3

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 26 April 2023 14.00 to 15.40

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 datasheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Figure 1 shows a bar of length L, moving with the initial velocity $v(x, 0) = v_0$ and striking a spring of stiffness k at time $t = 0$. The bar has a constant Young's modulus E, cross-section A and density ρ.

(a) Define the boundary and initial conditions of the bar. [20%]

(b) Derive the weak form of the equations governing the motion of the bar. [40%]

(c) Find the semi-discrete form of the weak form derived in part (b) using a single linear finite element. [40%]

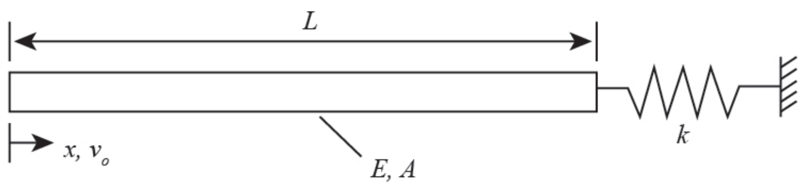

Fig. 1

2 (a) Figure 2(a) shows a finite element mesh consisting of one three-noded triangular element. The element represents a planar elastic body with Young's modulus $E = 200$ and Poisson's ratio $\nu = 0.3$ under plane strain condition. Consider only the unconstrained degrees of freedom.

(i) Determine the strain-displacement matrix $\boldsymbol{B}^e$. [20%]

(ii) Determine the stiffness matrix. [30%]

(iii) Determine the stiffness matrix when the element is uniformly scaled by a factor 1.5. [10%]

(b) Figure 2(b) shows a three-noded triangular element loaded with a force f_x and is supported by a two-noded one-dimensional bar element. The element's material parameters are as in part (a). The product of the Young's modulus and cross-sectional area is $EA = 100$. Consider only the unconstrained degrees of freedom.

(i) Determine the stiffness matrix of the bar element. [25%]

(ii) Determine the displacement of the node at which the load is applied. [15%]

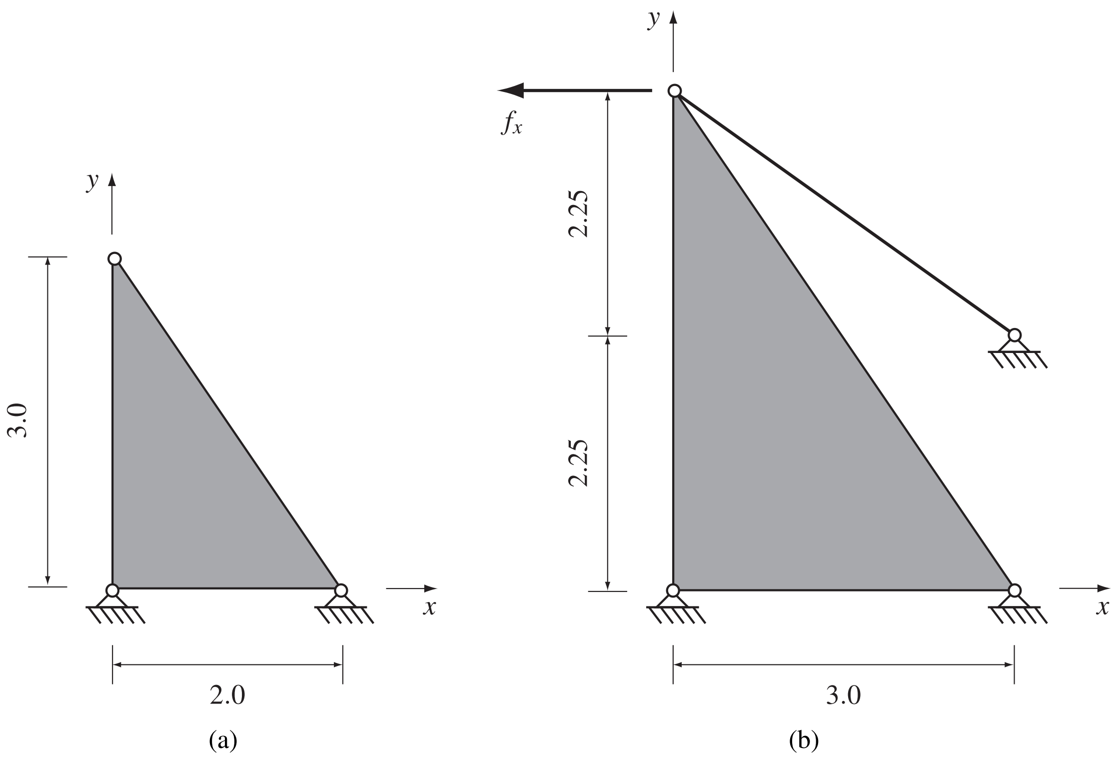

Fig. 2

3 (a) Figure 3 shows a four-noded isoparametric element in the physical domain and its parent element in the parametric domain.

(i) Using isoparametric mapping, give the expression for the physical coordinates of a point in terms of its coordinates in the parent element. [30%]

(ii) Calculate the coordinate of the Point P in the physical element corresponding to the point (0.5, 0.5) in the parent element. [10%]

(iii) Sketch the line to which the straight line between the nodes 1 and 3 in the parent element is transformed in the physical element. Is the mapped line straight? [15%]

(iv) Given a point Q with the coordinates (3, 1) in the physical element suggest a numerical technique for determining its coordinates in the parent element. No numerical computations are necessary. [15%]

(b) Consider on a two-dimensional domain the weak form

$$\int_\Omega (\nabla u)^T \nabla w \, d\Omega + \int_\Omega \boldsymbol{c}^T (\nabla u) w \, d\Omega + \int_\Omega f w \, d\Omega = 0 \, ,$$

where u is the unknown variable, w is the weight function, $\boldsymbol{c}$ is a prescribed vector and f is a prescribed function. The domain boundary consists of a Dirichlet and Neumann part, and on the Dirichlet part u is prescribed as zero. Determine the corresponding strong form. [30%]

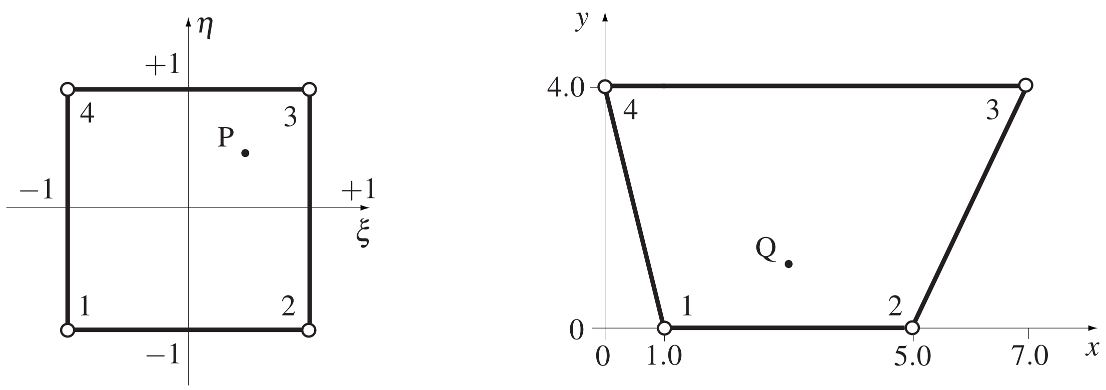

Fig. 3

4 (a) Describe what is meant by a priori and a posteriori error analysis. [20%]

(b) Describe two practical strategies that one can apply to reduce the finite element error with an a posteriori error estimator. [20%]

(c) A thin beam subjected to bending moments is modelled using beam elements of size h with polynomial order 3. Find an a priori error estimate for the deflection of the beam, measured in the L_2 norm. [30%]

(d) Consider the equation

$$\dot{y} + \lambda y = 0 \quad \text{with} \quad \lambda > 0$$

(i) Find an expression for the solution y_{n+1}, in terms of y_n, λ and time step size Δt using the Crank-Nicolson time stepping method.

(ii) Comment on the stability of the solution obtained using the Crank-Nicolson method. [30%]

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3D7 DATA SHEET

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\,\mathrm{d}V$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T\boldsymbol{f}\,\mathrm{d}V$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Heat flux	$\boldsymbol{q} = -\boldsymbol{D}\nabla T$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\,\mathrm{d}V$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI\boldsymbol{B}\,\mathrm{d}V$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

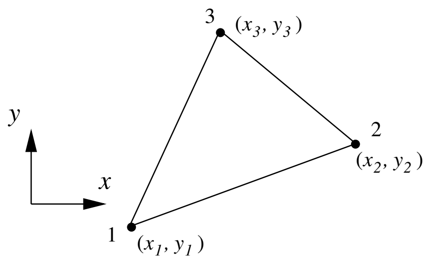
2D plane stress

$$\boldsymbol{D} = \frac{E}{\left(1-\nu^2\right)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D, isotropic)

$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

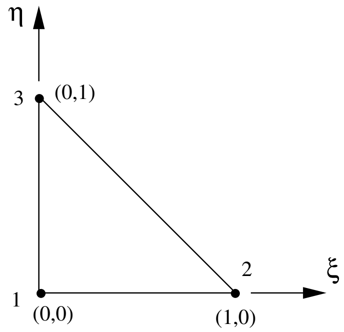
Shape functions



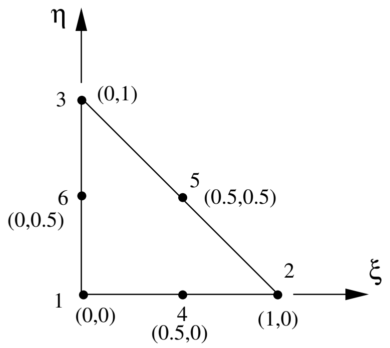
$$N_1 = ((x_2 y_3 - x_3 y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y)\,/2A$$
$$N_2 = ((x_3 y_1 - x_1 y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y)\,/2A$$
$$N_3 = ((x_1 y_2 - x_2 y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y)\,/2A$$

A = area of triangle

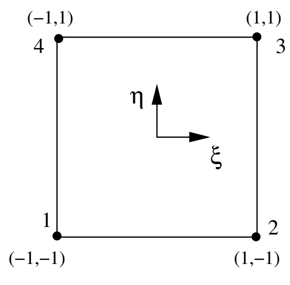
$$\int_\Omega N_i = A/3, \;\; i = 1,\ 2,\ 3.$$



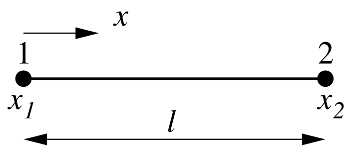
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\,(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\,(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\,(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)\,(1 - \eta)\,/4$$
$$N_2 = (1 + \xi)\,(1 - \eta)\,/4$$
$$N_3 = (1 + \xi)\,(1 + \eta)\,/4$$
$$N_4 = (1 - \xi)\,(1 + \eta)\,/4$$



Hermitian element

$$N_1 = \frac{-\left(x - x_2\right)^2 \left(-l + 2\left(x_1 - x\right)\right)}{l^3}$$

$$M_1 = \frac{\left(x - x_1\right)\left(x - x_2\right)^2}{l^2}$$

$$N_2 = \frac{\left(x - x_1\right)^2 \left(l + 2\left(x_2 - x\right)\right)}{l^3}$$

$$M_2 = \frac{\left(x - x_1\right)^2 \left(x - x_2\right)}{l^2}$$

Gauss integration in one dimension on the domain $(-1, 1)$

Using n Gauss integration points, a polynomial of degree $2n - 1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$