Question 1

(a) The procedure to derive weak formulation consists of three steps

1. Multiply the strong equation by a wegiht function which is equal to zero where Dirichlet boundary condition are applied but is otherwise arbitrary.
2. Use integration by parts to 'shift' derivatives to the weight function.
3. Insert the Neumann boundary condition

(b) First note

$$-(\frac{du}{dx})^2 - u\frac{d^2u}{dx^2} = -\frac{d}{dx}(u\frac{du}{dx}) \tag{1}$$

Following the three-step proceudre, we write the weak from:

$$\begin{aligned} 0 &= \int_0^1 v(-\frac{d}{dx}(u\frac{du}{dx}) + f)dx \qquad (2) \\ &= \int_0^1 (v\frac{dv}{dx}\frac{du}{dx} + vf)dx - (v(u\frac{du}{dx}))|_0^1, \qquad (3) \end{aligned}$$

where v is the test function. Plug in the boundary condition we obtain the weak formulation

$$0 = \int_0^1 (u\frac{du}{dx}\frac{dv}{dx} + vf)dx. \tag{4}$$

(c) The highest order of derivative in the weak formulation is 1, therefore, one can use a simple C^0 element to model the problem. The approximate solution u_h given by the C^0 element is continuous, but its first order derivative $\frac{du_h}{dx}$ is not continuous in general.

2) a)

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = -\frac{f}{D}$$

Consider first term:

$$\int \frac{\partial^4 u}{\partial x^4} w \, dx = \int \frac{\partial}{\partial x}\left(\frac{\partial^3 u}{\partial x^3} w\right) dx - \int \frac{\partial^3 u}{\partial x^3} \frac{\partial w}{\partial x} dx$$

$$= \int \frac{\partial}{\partial x}\left(\frac{\partial^3 u}{\partial x^3} w\right) dx - \int \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} \frac{\partial w}{\partial x}\right) + \int \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 w}{\partial x^2}$$

$\Longrightarrow$ Domain terms:

$$\int_\Omega \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 w}{\partial y^2}\right) = -\int_\Omega \frac{f}{D} w \, dx$$

Boundary terms:

$$\int_\Gamma \left(\frac{\partial^3 u}{\partial x^3} w + \frac{\partial^3 u}{\partial y^3} w\right) n \, dx \qquad n\text{-boundary normal}$$

$$\int \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial w}{\partial x} + \frac{\partial^2 u}{\partial y^2} \frac{\partial w}{\partial x}\right) n \, dx$$

b) Each boundary has two different boundary conditions because fourth order differential equation

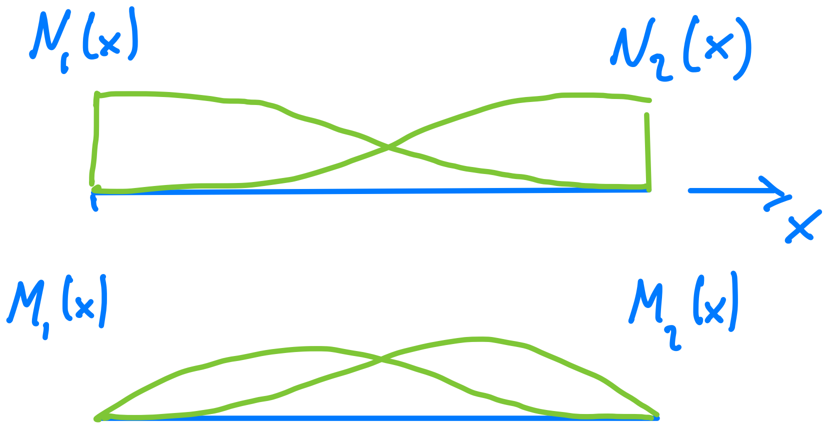
Displacements fixed: $w = 0$

Shear force prescribed: $\frac{\partial^3 w}{\partial x^3} \neq 0$, $\frac{\partial^3 w}{\partial y^3} \neq 0$

Slope fixed/clamped : $\frac{\partial w}{\partial x} = 0$

Moments prescribed : $\frac{\partial^2 w}{\partial x^2} \neq 0$, $\frac{\partial^2 w}{\partial y^2} \neq 0$

c) Tensor products of Hermite shape functions



$N_1(x,y) = N_1(x)\,N_1(y)$ $\qquad$ $M_1(x,y) = M_1(x)\,M_1(y)$

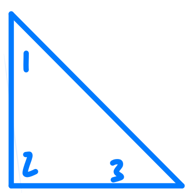
$N_2(x,y) = N_2(x)\,N_1(y)$ $\qquad$ $M_2(x,y) = M_2(x)\,M_1(y)$

$N_3(x,y) = N_2(x)\,N_2(y)$ $\qquad$ $M_3(x,y) = M_2(x)\,M_2(y)$

$N_4(x,y) = N_1(x)\,N_2(y)$ $\qquad$ $M_4(x,y) = M_1(x)\,M_2(y)$

d) We can use isoparametric mapping to obtain shape functions. However, we need to make sure that the obtained shape functions are indeed slope continuous at nodes where several elements meet.

3a)



$$\underline{f} = \int N(x)^T \begin{pmatrix} 0 \\ -1 \end{pmatrix} dx$$

$$= \frac{2}{3} \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix}^T$$

(Recall the volume of a tetrahedron)

b)

$$\begin{pmatrix} 0 \\ 4 \cdot \left(-\frac{2}{3}\right) \\ 0 \\ 4\left(-\frac{2}{3}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{8}{3} \\ 0 \\ -\frac{8}{3} \end{pmatrix}$$

c)

$$D = \begin{pmatrix} 200 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 100 \end{pmatrix}$$

$$\frac{\partial N_1}{\partial x} = 0 \qquad \frac{\partial N_1}{\partial y} = \frac{1}{2} \qquad \frac{\partial N_3}{\partial x} = \frac{1}{2} \qquad \frac{\partial N_3}{\partial y} = 0$$

$$\frac{\partial N_2}{\partial x} = -\frac{1}{2} \qquad \frac{\partial N_2}{\partial y} = -\frac{1}{2}$$

$$B = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$K = 2 \cdot B^T D B$$

$$= \begin{array}{c c} \begin{array}{cccccc} u_{x1} & u_{y1} & u_{x2} & u_{y2} & u_{x3} & u_{y3} \end{array} & \\ \begin{pmatrix} 25 & 0 & -25 & -25 & 0 & 25 \\ 0 & 50 & 0 & -50 & 0 & 0 \\ -25 & 0 & 75 & 25 & -50 & -25 \\ -25 & -50 & 25 & 75 & 0 & -25 \\ 0 & 0 & -50 & 0 & 50 & 0 \\ 25 & 0 & -25 & -25 & 0 & 25 \end{pmatrix} & \begin{array}{c} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{array} \end{array}$$

d)

$$\begin{array}{c} \begin{array}{cccc} u_{x2} & u_{y2} & u_{x3} & u_{y3} \end{array} \\ \begin{pmatrix} 3 \cdot 75 + 25 + 50 & 3 \cdot 25 & -2 \cdot 25 & 0 \\ 3 \cdot 25 & 3 \cdot 75 + 25 + 50 & -2 \cdot 25 & -2 \cdot 50 \\ -2 \cdot 25 & -2 \cdot 25 & 2(25+50) & 0 \\ 0 & -2 \cdot 50 & 0 & 2(25+50) \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 300 & 75 & -50 & 0 \\ 75 & 300 & -50 & -100 \\ -50 & -50 & 150 & 0 \\ 0 & -100 & 0 & 150 \end{pmatrix}$$

Question 4

(a) Let $[U_1, U_2]$ be the nodel displacement of the element, note from the boundary condition $U_1 = 0$. The semi-discretized form is given by:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ U_2 \end{bmatrix} + \frac{mAL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ \ddot{U}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ P \end{bmatrix} \tag{5}$$

which simplifies to

$$\ddot{U}_2 + \frac{3E}{mL^2}U_2 = \frac{3P_0}{mAL} \tag{6}$$

with $U_2(0) = 0$, $\dot{U}_2(0) = 0$.

(b) Assume the solution of 6 is of the form

$$U_2(t) = A\cos(\alpha t) + B\sin(\alpha t) + C \tag{7}$$

, with $\alpha = \sqrt{\frac{3E}{mL^2}}$.
From the initial condition we have:

$$U_2(t = 0) = A\cos(0) + B\sin(0) + C = 0 \tag{8}$$

$$\dot{U}_2(t = 0) = -A\alpha\sin(0) + B\alpha\cos(0) = 0 \tag{9}$$

Where we conclude $B = 0$ and $A = -C$.
Therefore U_2 simplifies to

$$U_2(t) = -C\cos(\alpha t) + C \tag{10}$$

To obtain the value of C we substitute the above equation to the governing equation:

$$-\alpha^2 C\cos(\alpha t) + \frac{3E}{mL^2}(-C\cos(\alpha t) + C) = \frac{3P_0}{mAL} \tag{11}$$

Note $\alpha^2 = \frac{3E}{mL^2}$, then,

$$\frac{3E}{mL^2}(C) = \frac{3P_0}{mAL} \tag{12}$$

Therefore

$$C = \frac{P_0 L}{EA} \tag{13}$$

The displacement field in the bar is

$$u(x,t) = U_1(-\frac{1}{L}x + 1) + U_2\frac{x}{L}. \tag{14}$$

With $U_1 = 0$, we conclude

$$u(x,t) = U_2\frac{x}{L} = \frac{P_0}{EA}(1 - \cos(\sqrt{\frac{3E}{mL^2}}t)) \cdot x. \tag{15}$$