Version FC/4

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 24 April 2024   14.00 to 15.40

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 datasheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Define the procedure to derive the weak formulation of a partial differential equation. [20%]

(b) A nonlinear boundary value problem is given by

$$-\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - u\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + f = 0 \quad \text{for } 0 \le x \le 1$$

where u is the unknown solution field and f a prescribed forcing. For the boundary conditions $\left.\left(u\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right|_{x=0} = 0$, and $u(x=1) = \sqrt{2}$, derive its weak formulation. [60%]

(c) What is the simplest kind of element that one may use to discretise the weak form derived in part (b)? Comment on the continuity of the respective finite element solution. [20%]

2 Consider the bending of an orthotropic thin plate described by the equilibrium equation

$$\frac{\partial^4 u(x,y)}{\partial x^4} + \frac{\partial^4 u(x,y)}{\partial y^4} = -\frac{f(x,y)}{D}$$

where $u(x,y)$ is the deflection of the plate, $f(x,y)$ is the distributed pressure loading and D is the constant flexural rigidity.

(a) Derive the weak form of the thin plate equilibrium equation. [35%]

(b) Identify the relevant boundary conditions in the derived weak form and briefly comment on their choice in typical structural engineering problems. [25%]

(c) Propose suitable shape functions for the square thin plate finite element shown in Fig. 1. [25%]

(d) Briefly comment on how to derive shape functions for arbitrarily shaped quadrilateral thin plate finite elements. [15%]

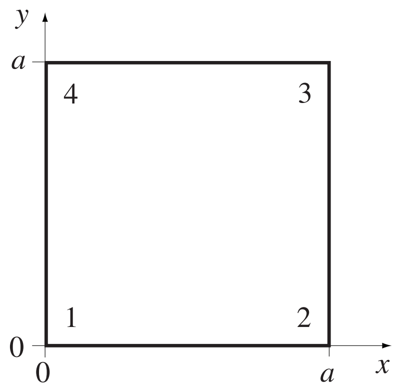

Fig. 1

3 Figure 2(a) shows a square planar elastic domain with Young's modulus $E = 200$ and Poisson's ratio $\nu = 0$. Along the left and right domain boundaries the displacements are prescribed to be zero. Throughout the domain is a distributed body force $g = 1$ acting in the negative y-direction. The domain is discretised with three-noded triangular elements, an example of which is shown in Fig. 2(b).

(a) Compute the element external force vector of the three-noded triangular element in Fig. 2(b). [20%]

(b) Compute the global external force vector components corresponding to the degrees of freedom of nodes 2 and 3 in Fig. 2(b). [10%]

(c) Compute the element stiffness matrix components corresponding to the degrees of freedom of nodes 1 and 2 in Fig. 2(a). This matrix will be of dimension 4×4. [45%]

(d) Compute the global stiffness matrix components corresponding to the degrees of freedom of nodes 2 and 3 in Fig. 2(a). This matrix will be of dimension 4×4. [25%]

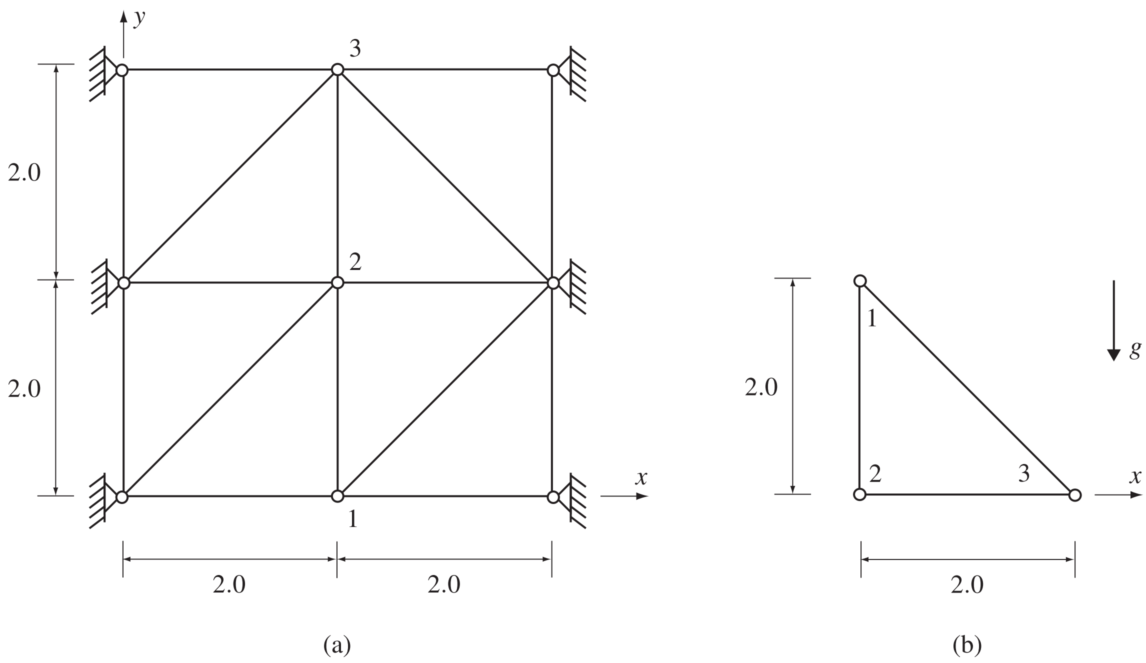

Fig. 2

4 Consider a bar with a uniform cross-section area A, Young's modulus E, mass density per unit length m, and length L. The bar is fixed at $x = 0$, and subjected to a time-dependent axial force $P(t)$ at $x = L$. The axial displacement $u(x, t)$ under the action force $P(t)$ is governed by the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \left(\frac{E}{m}\right) \frac{\partial^2 u(x,t)}{\partial x^2}$$

(a) Derive the semi-discrete form of the wave equation. Discretise the problem with one linear finite element and use a consistent mass matrix. [50%]

(b) Building on the result of part (a), find an expression of the displacement field $u(x, t)$ in the bar. [50%]

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3D7 DATA SHEET

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\,\mathrm{d}V$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\,\mathrm{d}V$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Heat flux	$\boldsymbol{q} = -\boldsymbol{D}\nabla T$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\,\mathrm{d}V$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI\boldsymbol{B}\,\mathrm{d}V$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

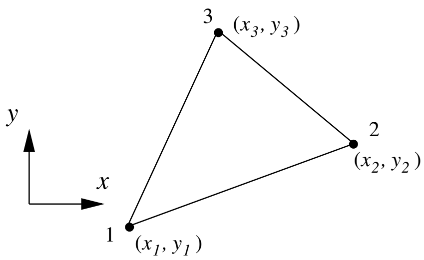
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D, isotropic)

$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

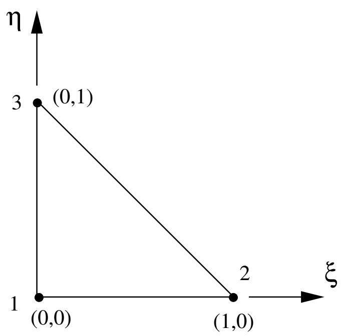
Shape functions



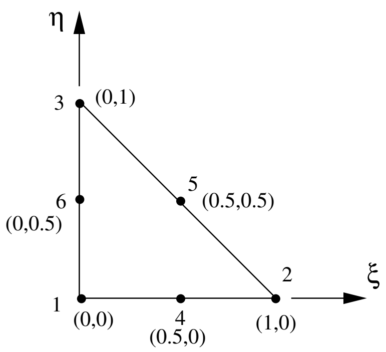
$$N_1 = ((x_2y_3 - x_3y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y)\,/2A$$
$$N_2 = ((x_3y_1 - x_1y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y)\,/2A$$
$$N_3 = ((x_1y_2 - x_2y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y)\,/2A$$

A = area of triangle

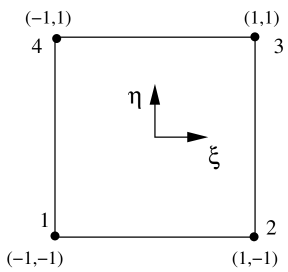
$$\int_\Omega N_i = A/3,\ \ i = 1,\ 2,\ 3.$$



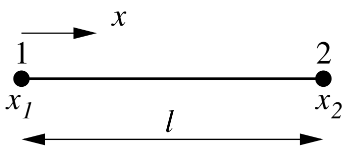
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\,(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\,(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\,(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)\,(1 - \eta)\,/4$$
$$N_2 = (1 + \xi)\,(1 - \eta)\,/4$$
$$N_3 = (1 + \xi)\,(1 + \eta)\,/4$$
$$N_4 = (1 - \xi)\,(1 + \eta)\,/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1, 1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$