Version FC/2

EGT2
ENGINEERING TRIPOS PART IIA

Monday 12 May 2025 2 to 3.40

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 datasheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Consider a simply supported beam of unit length lying on an elastic foundation. Its deflection $w(x)$ satisfies the differential equation

$$\frac{d^4 w(x)}{dx^4} + w(x) = 1$$

for $x \in (0, 1)$. The boundary conditions of the beam are

$$w(0) = 0 \qquad w(1) = 0 \qquad \frac{d^2 w}{dx^2}(0) = 0 \qquad \frac{d^2 w}{dx^2}(1) = 0$$

(a) Derive the weak formulation of this problem. [40%]

(b) The deflection of the beam can be modelled by a trial solution

$$w(x) = c_1 \sin(\pi x) + c_2 \sin(3\pi x) + c_3$$

Find the values of c_1, c_2 and c_3. [60%]

2 (a) While running a finite element software programme, it stops with the error: `Stiffness matrix is singular`. Give three possible reasons for this error message and describe how you would resolve each. [10%]

(b) A cantilever plate with tip loading is analysed using finite elements. The obtained boundary reaction forces are not in equilibrium with the applied tip loading. Give the main reason for this discrepancy and suggest how it can be reduced. [10%]

(c) What are the advantages of Gaussian quadrature over other common numerical integration techniques, and why is it widely used in finite elements? [5%]

(d) Figure 1 shows a four-noded isoparametric element in the physical domain and its parent element in the parametric domain.

(i) Determine the isoparametric mapping for this element. [25%]

(ii) Compute the Jacobian matrix of the element. [20%]

(iii) Compute the approximate nodal source vector for the source given by

$$s(x, y) = e^{xy}$$

using a single Gauss integration point. Briefly comment on the accuracy of the obtained result and how it can be improved. [30%]

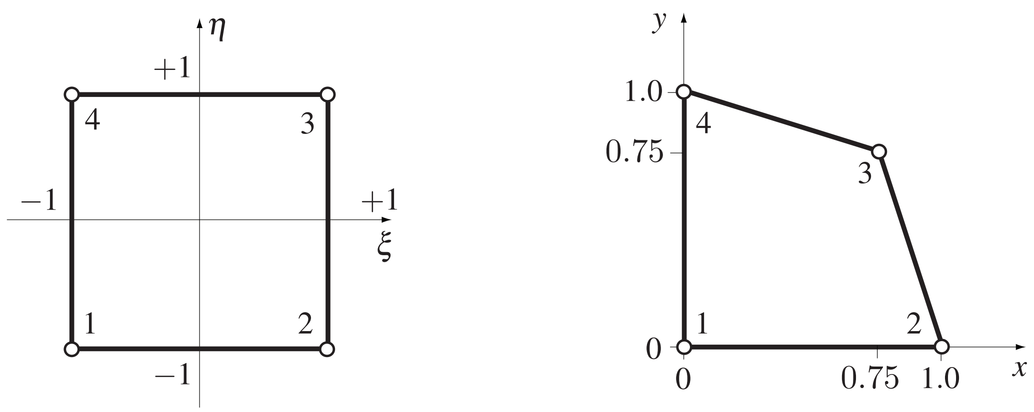


Fig. 1

3 (a) Determine and sketch the shape functions for the nodes 1, 2 and 3 of the three-noded finite element shown in Fig. 2(a). [5%]

(b) Determine the shape functions for the nodes 3, 9 and 8 of the nine-noded finite element shown in Fig. 2(b). [15%]

(c) Give two reasons why a nine-noded quadrilateral element might be preferred over a four-noded quadrilateral element. [10%]

(d) Consider on the domain Ω with the boundary Γ the two-dimensional problem governed by the differential equation

$$\boldsymbol{\alpha} \cdot \nabla u - \nabla \cdot (\beta \nabla u) = -f$$

where u is the temperature field and f is the source field. The prescribed vector $\boldsymbol{\alpha}$ and the positive scalar β are constant.

(i) Derive the weak form of this problem accounting for both Dirichlet and Neumann boundary conditions. [30%]

(ii) Determine the stiffness matrix of the three-noded finite element shown in Fig. 2(c). [40%]

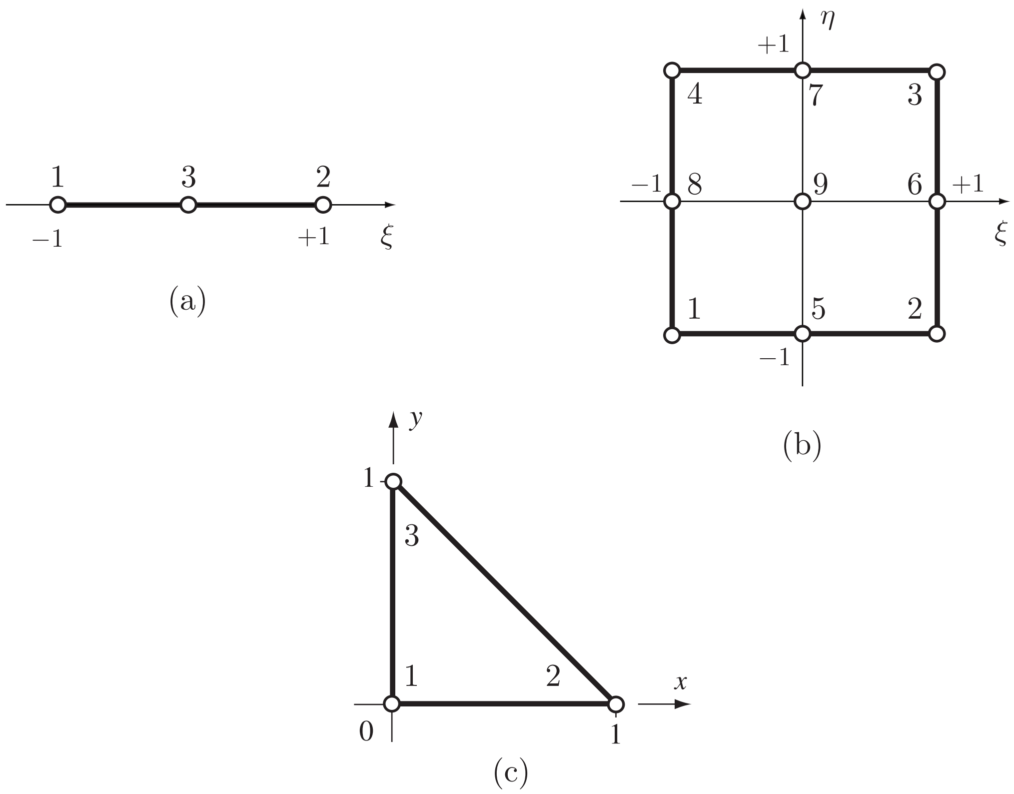


Fig. 2

4 Consider the semi-discrete finite element equation

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$$

where $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{K}$ the stiffness matrix and $\boldsymbol{f}$ the prescribed load vector.

(a) Using the average acceleration scheme

$$\begin{aligned} y_{n+1} &= y_n + \Delta t\, \dot{y}_n + \tfrac{\Delta t^2}{4} \left(\ddot{y}_n + \ddot{y}_{n+1}\right) \\ \dot{y}_{n+1} &= \dot{y}_n + \tfrac{\Delta t}{2} \left(\ddot{y}_n + \ddot{y}_{n+1}\right) \end{aligned}$$

Derive a fully discrete version of the given semi-discrete finite element equation. [60%]

(b) Is the average acceleration scheme an explicit or an implicit scheme? Briefly comment on the stability and accuracy properties of the average acceleration scheme. [20%]

(c) Why is a lumped mass matrix appropriate for use with an (at least partially) explicit or semi-explicit scheme? How is a lumped mass matrix typically constructed from the consistent mass matrix? [20%]

END OF PAPER

3D7 DATA SHEET

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \, \mathrm{d}V$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f} \, \mathrm{d}V$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Heat flux	$\boldsymbol{q} = -\boldsymbol{D}\nabla T$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \, \mathrm{d}V$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI \boldsymbol{B} \, \mathrm{d}V$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

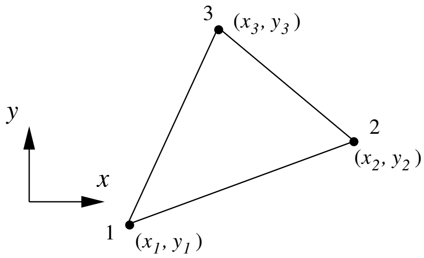
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D, isotropic)

$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

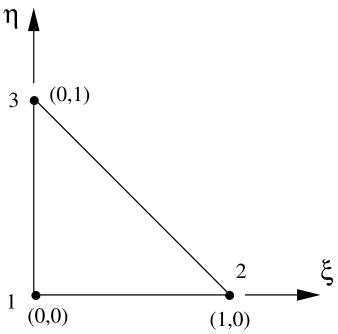
Shape functions



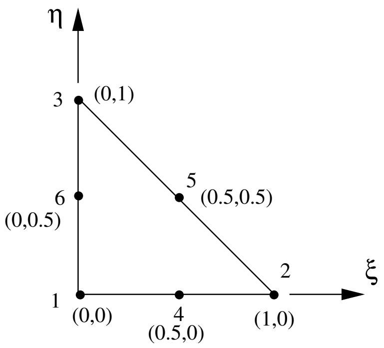
$$N_1 = ((x_2 y_3 - x_3 y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y)\,/2A$$
$$N_2 = ((x_3 y_1 - x_1 y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y)\,/2A$$
$$N_3 = ((x_1 y_2 - x_2 y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y)\,/2A$$

A = area of triangle

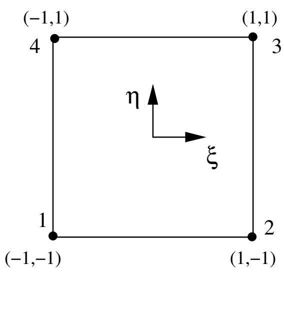
$\int_\Omega N_i = A/3, \; i = 1, 2, 3.$



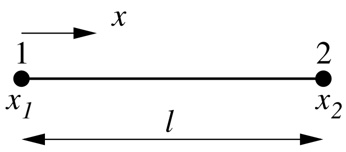
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\,(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\,(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\,(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)\,(1 - \eta)\,/4$$
$$N_2 = (1 + \xi)\,(1 - \eta)\,/4$$
$$N_3 = (1 + \xi)\,(1 + \eta)\,/4$$
$$N_4 = (1 - \xi)\,(1 + \eta)\,/4$$



Hermitian element

$$N_1 = \frac{-(x - x_2)^2\,(-l + 2\,(x_1 - x))}{l^3}$$

$$M_1 = \frac{(x - x_1)\,(x - x_2)^2}{l^2}$$

$$N_2 = \frac{(x - x_1)^2\,(l + 2\,(x_2 - x))}{l^3}$$

$$M_2 = \frac{(x - x_1)^2\,(x - x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1, 1)$

Using n Gauss integration points, a polynomial of degree $2n - 1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$