3D7: Finite Element Method
— Crib for 2014 —

1. (a) Strong form

$$\dot{c} = (Dc')'$$

Weak form

$$\int_\Omega \dot{c}w = \int_\Omega (Dc')'w = \int_\Omega (Dc'w)' - \int_\Omega Dc'w'$$
$$\int_\Omega \dot{c}w = [Dc'w]_0^L - \int_\Omega Dc'w'$$

$$\Rightarrow \int_\Omega \dot{c}w + \int_\Omega Dc'w' + j_0 w(L) = 0$$

(b) i.

$$\dot{c} = (Dc')' - \overline{D}(c\sigma')'$$

First consider the contribution of the last term

$$-\int_\Omega \overline{D}(c\sigma')'w = -\overline{D}\int_\Omega (c\sigma'w)' + \overline{D}\int_\Omega c\sigma'w'$$
$$= -\overline{D}[c\sigma'w]_0^L + \overline{D}\int_\Omega c\sigma'w'$$

Adding this to the weak form obtained in (a) gives

$$\int_\Omega \dot{c}w + \int_\Omega Dc'w' - \overline{D}\int_\Omega c\sigma'w' + (j_0 + \overline{D}c\sigma')w(L) = 0$$

At Dirichlet boundaries always $w(0) = 0$!

ii.

$$\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$$

The shape functions of an element with length h read

$$N_1(x) = 1 - \frac{x}{h} \qquad N_2(x) = \frac{x}{h}$$

This gives a mass matrix $\boldsymbol{M}$

$$M_{ij} = \int_0^h N_i(x)N_j(x) = h\begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

Following two integrals contribute to the stiffness matrix $\boldsymbol{K}$

$$\int_\Omega Dc'w' \quad \text{and} \quad -\overline{D}\int_\Omega c\sigma'w'$$

For given nodal stress values (σ_1, σ_2) and linear shape functions the stress derivative is constant

$$\sigma' = \frac{\sigma_2 - \sigma_1}{h}$$

Hence, the element stiffness matrix $\boldsymbol{K}$ is obtained from

$$K_{ij} = D \int_0^h N_i' N_j' - \overline{D} \frac{\sigma_2 - \sigma_1}{h} \int_0^h N_i' N_j$$

Inserting the linear shape functions and integrating yields

$$D \int_0^h N_i' N_j' = \frac{D}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$-\overline{D} \frac{\sigma_2 - \sigma_1}{h} \int_0^h N_i' N_j = -\overline{D} \frac{\sigma_2 - \sigma_1}{h} \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The element stiffness matrix is the sum of the two matrices

$$K_{ij} = \frac{D}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \overline{D} \frac{\sigma_2 - \sigma_1}{h} \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Notice that this matrix is unsymmetric. The correct sign of the off-diagonal terms is important!

The last element in the mesh contributes to the force vector $\boldsymbol{f}$

$$f_i = \begin{pmatrix} 0 \\ -j_0 \end{pmatrix}$$

Also, the last element contributes, in addition to the above, the following to the stiffness matrix

$$\overline{D} \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_2 - \sigma_1}{h} \end{pmatrix}$$

iii. When the diffusion coefficient is a function of the concentration, the finite element problem is nonlinear. This means that the stiffness matrix coefficients are a function of the unknown nodal concentrations.

For explicit time integration the nonlinearity of the semi-discrete equations is not relevant.

$$\boldsymbol{M} \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{\Delta t} = \boldsymbol{f}_n - \boldsymbol{K}(\boldsymbol{u}_n)\boldsymbol{u}_n$$

This equation is linear in $\boldsymbol{u}_{n+1}$.

However, for implicit time integration the equations are nonlinear in $\boldsymbol{u}_{n+1}$

$$\boldsymbol{M} \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{\Delta t} + \boldsymbol{K}(\boldsymbol{u}_{n+1})\boldsymbol{u}_{n+1} = \boldsymbol{f}_{n+1}$$

The solution can only be obtained iteratively. For this one of the common root-finding algorithms, such as bisectioning or Newton's method, are used.

2. (a) i. The three shape functions of the element are

$$N_1 = 1 - \frac{x}{2} - \frac{y}{2}$$
$$N_2 = \frac{x}{2}$$
$$N_3 = \frac{y}{2}$$

ii. The strain-displacement relationship reads

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{pmatrix} = \underbrace{\begin{pmatrix} N_{2,x} & 0 \\ 0 & N_{2,y} \\ N_{2,y} & N_{2,x} \end{pmatrix}}_{\boldsymbol{B}^e} \begin{pmatrix} u_{2x} \\ u_{2y} \end{pmatrix}$$

$N_{2,x}$ means differentiation with respect to x etc.

$$\Rightarrow \boldsymbol{B}^e = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

As to be expected the $\boldsymbol{B}^e$ matrix for this linear element is constant.

iii. The stiffness-matrix can be computed without numerical integration

$$\boldsymbol{K}^e = \int_\Omega \boldsymbol{B}^{eT}\boldsymbol{D}\boldsymbol{B} = 2\boldsymbol{B}^{eT}\boldsymbol{D}\boldsymbol{B}$$

with

$$\boldsymbol{D} = \begin{pmatrix} 200 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 100 \end{pmatrix}$$

This yields for the stiffness matrix

$$\boldsymbol{K}^e = \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix}$$

(b) i. The global stiffness matrix of the two element mesh is assembled by inspection

$$\boldsymbol{K} = \begin{pmatrix} 100 + 50 & 0 \\ 0 & 50 + 100 \end{pmatrix}$$

ii. The discrete equilibrium equations in the global $x - y$ coordinate system read

$$\begin{pmatrix} 150 & 0 \\ 0 & 150 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

Now consider the displacements in a local coordinate system oriented with the inclined roller support

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Introducing this into the equilibrium equation in the global $x - y$ coordinate system leads to

$$150 \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_x \\ 0 \end{pmatrix}$$

The roller constrains $u_2 = 0$ hence we obtain from the first equation

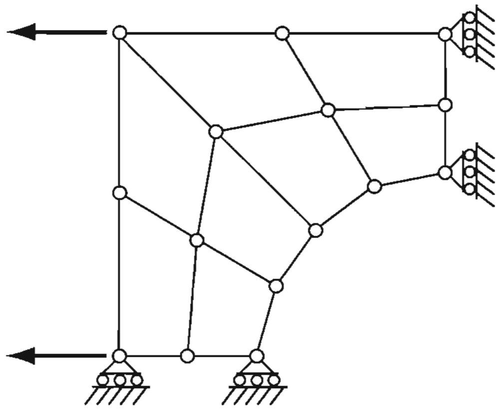
$$u_1 = \frac{f_x}{150 \cos 30°} \approx 0.00769$$

3. (a) In element 3 close to the non-convex corner the determinant of the Jacobian is negative. The integration of element integrals will yield wrong values. For instance, if we would compute the area of the element

$$\int_\Omega d\Omega = \int_{-1}^{+1}\int_{-1}^{+1} |\boldsymbol{J}^e(\xi,\eta)|\, d\xi d\eta$$

it would give a wrong value. Moreover, the global stiffness matrix for this mesh is rank deficient and the matrix is not invertible.

A better mesh would be



(b) i. The Jacobian J is computed with the isoparametric mapping

$$x = \sum_i N_i x_i \qquad y = \sum_i N_i y_i$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

The four shape functions of the element

$$\begin{aligned} N_1 &= (1-\xi)(1-\eta)/4 \\ N_2 &= (1+\eta)(1-\eta)/4 \\ N_3 &= (1+\xi)(1+\eta)/4 \\ N_4 &= (1-\xi)(1+\eta)/4 \end{aligned}$$

yield the Jacobian matrix

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{1-\eta}{4}4 + \frac{1+\eta}{4}8 - \frac{1+\eta}{4}4 = 2 \\ \frac{\partial x}{\partial \eta} &= -\frac{1+\xi}{4}4 + \frac{1+\xi}{4}8 + \frac{1-\xi}{4}4 = 2 \\ \frac{\partial y}{\partial \xi} &= \frac{1+\eta}{4}6 - \frac{1+\eta}{4}6 = 0 \\ \frac{\partial y}{\partial \eta} &= \frac{1+\xi}{4}6 + \frac{1-\xi}{4}6 = 3 \end{aligned}$$

$$\Rightarrow J = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$$

ii. The strain components ϵ_{xx} and ϵ_{yy} follow from

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x}$$
$$\epsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y}$$

The displacements are interpolated with

$$u_x = 0.1 \cdot N_4 \qquad u_y = 0.2 \cdot N_4$$

and the inverse of the Jacobian reads

$$J^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Introducing the displacement interpolation and the inverse of the Jacobian into the strain equations gives

$$\epsilon_{xx} = -\frac{1+\eta}{80}$$
$$\epsilon_{yy} = \frac{1}{60}(2+\eta-\xi)$$

(c) The scalar field u is assumed as linear in the tetrahedron

$$u = \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4$$

where α_1, α_1, α_2 and α_3 are four unknowns. The scalar field u can also be expressed as a function of the four nodal values u_1, u_2, u_3 and u_4 and the corresponding shape functions

$$u = N_1(x,y,z)u_1 + N_2(x,y,z)u_2 + N_3(x,y,z)u_3 + N_4(x,y,z)u_4$$

The shape function N_i has the value 1 at node i and is 0 at every other node.
For instance, the coefficients α_1, α_1, α_2 and α_3 defining the shape function N_1 can be computed by solving

$$\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

4. (a) The integration scheme

$$\dot{\boldsymbol{a}}_n = \frac{\boldsymbol{a}_{n+1} - \boldsymbol{a}_{n-1}}{2\Delta t}$$

leads to the accelerations

$$\ddot{\boldsymbol{a}}_n = \frac{\dot{\boldsymbol{a}}_{n+1} - \dot{\boldsymbol{a}}_{n-1}}{2\Delta t} = \frac{\boldsymbol{a}_{n+2} - 2\boldsymbol{a}_n + \boldsymbol{a}_{n-2}}{4\Delta t^2}$$

The equilibrium equations at time step n are considered

$$\boldsymbol{M}\ddot{\boldsymbol{a}}_n + \boldsymbol{K}\boldsymbol{a}_n = \boldsymbol{f}_n$$

Introducing the accelerations

$$\boldsymbol{M}\frac{\boldsymbol{a}_{n+2} - 2\boldsymbol{a}_n + \boldsymbol{a}_{n-2}}{4\Delta t^2} + \boldsymbol{K}\boldsymbol{a}_n = \boldsymbol{f}_n$$

and (decrementing the indices by one) yields

$$\boldsymbol{M}\boldsymbol{a}_{n+1} = 4\Delta t^2(\boldsymbol{f}_{n-1} - \boldsymbol{K}\boldsymbol{a}_{n-1}) + 2\boldsymbol{M}\boldsymbol{a}_{n-1} - \boldsymbol{M}\boldsymbol{a}_{n-3}$$

(b) Explicit scheme. Time step size has to be chosen small otherwise the solution will be unstable. This means the displacements, velocities and accelerations will become arbitrarily large.

(c) Need maximum eigenvalue of

$$(\boldsymbol{K} - \omega^2 \boldsymbol{M})\phi = 0$$

The stiffness matrix and the mass matrix for a linear bar element are

$$\boldsymbol{K} = \frac{EA}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\boldsymbol{M} = \rho A h \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Here, a lumped mass matrix is used. It is also possible to use the consistent mass matrix. The maximum eigenvalue is obtained from

$$\det \begin{pmatrix} \frac{E}{h} - \omega^2 \frac{\rho h}{2} & -\frac{E}{h} \\ -\frac{E}{h} & \frac{E}{h} - \omega^2 \frac{\rho h}{2} \end{pmatrix} = 0$$

$$\Rightarrow \omega_{\text{max}} = \frac{2}{h}\sqrt{\frac{E}{\rho}} = \frac{2c}{h}$$

where $c^2 = E/h$ is the longitudinal elastic wave speed. For stability the time step size Δt has to be such that

$$\Delta t \leq \frac{2}{\omega_{\text{max}}} = \frac{h}{c}$$

(d) Cubic element has four nodes. Distance between nodes is $h/3$. Time step size should reduce by about $1/3$. The propertionality is still h/c.