Q.1

a-

(i) The shape function for Node 3 is

$$N^3 = \frac{y}{2L}$$

The stiffness matrix in plain strain elasticity writes

$$K = (B^e)^T D\, B^e A^e \qquad (B^e \text{ is constant over the element})$$

where $$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

For node 3

$$B^e_3 = \begin{bmatrix} \frac{\partial N^3}{\partial x} & 0 \\ 0 & \frac{\partial N^3}{\partial y} \\ \frac{\partial N^3}{\partial y} & \frac{\partial N^3}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2L \\ 1/2L & 0 \end{bmatrix}$$

$$(B^e_3)^T D\, (B^e_3) = \begin{bmatrix} 0 & 0 & 1/2L \\ 0 & 1/2L & 0 \end{bmatrix} \begin{bmatrix} 0 & \nu/2L \\ 0 & (1-\nu)/2L \\ \frac{(1-2\nu)}{4L} & 0 \end{bmatrix} \frac{E}{(1+\nu)(1-2\nu)}$$

$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} \frac{(1-2\nu)}{2}\frac{1}{4L^2} & 0 \\ 0 & (1-\nu)\frac{1}{4L^2} \end{bmatrix}$$

$$A^e = L^2 \quad \Rightarrow \quad K^3 = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} \frac{(1-2\nu)}{8} & 0 \\ 0 & \frac{(1-\nu)}{4} \end{bmatrix}$$

(ii) $$f_e^3 = -\rho_s g \begin{bmatrix} 0 \\ A^e/3 \end{bmatrix} + \rho_w g \int_0^{2L} (2L-y)\, N^3(0,y)\, dy \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$f_e^3 = -\rho_s g \frac{L^2}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \rho_w g \left[\frac{y^2}{2} - \frac{1}{2L}\frac{y^3}{3} \right]_0^{2L} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$f_e^3 = -\rho_s g \frac{L^2}{3}\begin{bmatrix}0\\1\end{bmatrix} + \frac{\rho_w g (2L)^2}{6}\begin{bmatrix}1\\0\end{bmatrix}$$

(iii) No. The mesh should be refined.

b - i $\quad x = \begin{bmatrix}N^1 & N^2 & N^3 & N^4 & N^5 & N^6\end{bmatrix}\begin{bmatrix}0\\L\\0\\L/2\\d\\0\end{bmatrix}$

$$x = \xi L + \xi\eta(4d - 2L)$$

$$y = \begin{bmatrix}N^1 & N^2 & N^3 & N^4 & N^5 & N^6\end{bmatrix}\begin{bmatrix}0\\0\\2L\\0\\L\\L\end{bmatrix}$$

$$y = 2L\eta$$

(ii) $\quad J^e = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix} = \begin{bmatrix}L + \eta(4d-2L) & 0\\ \xi(4d-2L) & 2L\end{bmatrix}$

Stiffness matrix:

$$K^e = \sum_{I=1}^{N}\sum_{J=1}^{N}\left(B^e(\xi_I, \eta_J)\right)^T D\, B^e(\xi_I, \eta_J)\, |J^e(\xi_I, \eta_J)|\, \omega_I \omega_J$$

and $\begin{pmatrix}\partial N^i/\partial x\\ \partial N^i/\partial y\end{pmatrix} = \left(J^e\right)^{-1}\begin{pmatrix}\partial N^i/\partial \xi\\ \partial N^i/\partial \eta\end{pmatrix}$ to compute B-matrix

(iii) $\det J^e = 2L(L + \eta(4d - 2L))$

$d = \frac{L}{4} \Rightarrow \det J^e = 1 - \eta$

for $\eta = 1$

$\det J^e = 0$!! Jacobian is singular

Question 2

a.

$$\int w \frac{\mathrm{d}^4 u}{\mathrm{d}x^4}\,\mathrm{d}x = \int w\, f\,\mathrm{d}x$$

Integrate by parts once,

$$-\int \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3}\,\mathrm{d}x + \left[w\frac{\mathrm{d}^3 u}{\mathrm{d}x^3}\right] = \int w\, f\,\mathrm{d}x.$$

Integrate by parts a second time

$$\int \frac{\mathrm{d}^2 w}{\mathrm{d}^2 x}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,\mathrm{d}x - \left[\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right] + \left[w\frac{\mathrm{d}^3 u}{\mathrm{d}x^3}\right] = \int w\, f\,\mathrm{d}x.$$

At boundary, $u = 0$ and so $w = 0$, and $\frac{\mathrm{d}^2 u}{\mathrm{d}^2 x} = 0$.

As a result

$$\int \frac{\mathrm{d}^2 w}{\mathrm{d}^2 x}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,\mathrm{d}x = \int w\, f\,\mathrm{d}x.$$

b.

Hermite $\rightarrow$ C^1-continuous

c.

Lowest order per node: 2 dofs

4 unknowns (2 nodes) $\rightarrow$ cubic shape functions

Integrands:

- Left-hand side: the integrand is a product of linear shape functions $\rightarrow$ quadratic

Use 2 Gauss points $\rightarrow$ integrate exactly

- Right-hand side: the integrand is cubic $\rightarrow$ 2 Gauss points as well

d.

$$\|u - u_h\|_k \leq c h^{p+1-k} \|u\|$$

$p = 3$ (cubic), $k = 1$ $\rightarrow h^3$

Halve the element size $\rightarrow$ error reduced by $(1/2)^3 = 1/8$

e.

Two coupled equations:

$$-\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = f \qquad (1)$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + v = 0 \qquad (2)$$

Week forms (ignoring boundaries)

$$\int \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int w\,f\,\mathrm{d}x \qquad (1)$$

$$-\int \frac{\mathrm{d}q}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x + \int q v\,\mathrm{d}x = 0 \qquad (2)$$

The second term in the left-hand side of equation (2) corresponds to the standard mass matrix $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} l/3 & l/6 \\ l/6 & l/3 \end{pmatrix}.$$

The system writes in matrix form

$$\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{M} & -\mathbf{K} \end{pmatrix}\begin{pmatrix} \underline{v} \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

where $\mathbf{K}$ is the standard stiffness matrix,

$$\mathbf{K} = \begin{pmatrix} 1/l & -1/l \\ -1/l & 1/l \end{pmatrix}.$$

Splitting the equation in this case opens the possibility of using simple Lagrange shape functions.

Qu 3)

a) Shape functions must be compatible across element boundaries. This is only possible when ansatz is linear for 3 nodes.

b)

i) $N_1 = 1 - \frac{x}{2} - \frac{y}{2}$

$N_2 = \frac{x}{2}$

$N_3 = \frac{y}{2}$

$$f = s \int N \, d\Omega \Rightarrow f^e = \frac{2 \cdot A_e}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

ii) $$\begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{pmatrix}}_{B} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

$\frac{\partial N_1}{\partial x} = -\frac{1}{2}$ $\quad \frac{\partial N_1}{\partial y} = -\frac{1}{2}$

$\frac{\partial N_2}{\partial x} = \frac{1}{2}$ $\quad \frac{\partial N_2}{\partial y} = 0$

$\frac{\partial N_3}{\partial x} = 0$ $\quad \frac{\partial N_3}{\partial y} = \frac{1}{2}$

$$K = A \times 3 \cdot \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} & 0 \\ -\frac{3}{2} & 0 & \frac{3}{2} \end{pmatrix}$$

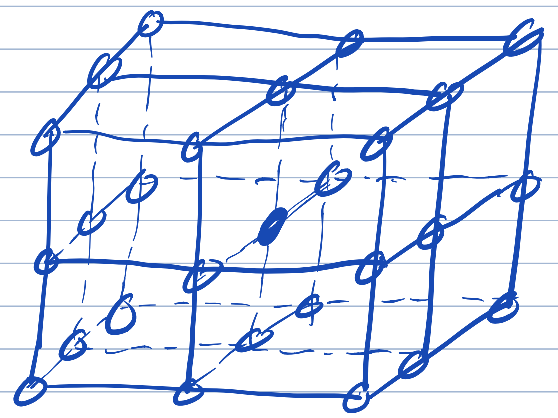
$A = 2$

iii) For the centre node

$$8 \cdot \frac{3}{2} T = 8 \cdot \frac{4}{3} \Rightarrow T = \frac{8}{9}$$

Q4

a) Count the number of 'connections' for a node surrounded by cells, plus 1 for the node itself:



9 nodes in each layer

$\Rightarrow$ $9 \times 3 = 27$ connected nodes

For elasticity there are 3 dofs per node

$\Rightarrow$ $27 \times 3 = 81$ non-zeros per matrix row

Same for case A and case B

b) Double number of elements $\Rightarrow$ double assembly time

Number of cells $m = (n_0 - 1)^3$, where n_0 is number of nodes/vertices in each direction

For large n_0

$$m \simeq n_0^3 = n,$$ where n is the number of nodes

$\uparrow$ #cells

memory cost $\propto n \Rightarrow$ memory cost doubles when doubling # cells

c) Number of dofs is proportional to the number of cells

LU: $O(n^2) \rightarrow O((2n)^2) = 4O(n^2)$ $\therefore$ cost increases factor 4

MG: $O(n) \rightarrow O(2n)$ $\therefore$ cost increases factor 2

d) $e = \left(\int_\Omega \left(\varepsilon_{ij} - \varepsilon^h_{ij}\right)\left(\varepsilon_{ij} - \varepsilon^h_{ij}\right) d\Omega \right)^{\frac{1}{2}}$

where $\varepsilon_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) / 2$ | u: exact soln

$\varepsilon^h_{ij} = \left(\frac{\partial u^h_i}{\partial x_j} + \frac{\partial u^h_j}{\partial x_i} \right) / 2$ | u^h: FE soln

e) Measure of cell size (length) $h \propto 1/n_0$

Where n_0 is number of cells in each dir.

$$n_0 = \underbrace{\sqrt[3]{n}}_{\#\text{nodes}} = \underbrace{\sqrt[3]{m}}_{\#\text{cells}}$$

$n_0^{(A)} = \sqrt[3]{m}$ (mesh A)

$n_0^{(B)} = \sqrt[3]{2m} = \sqrt[3]{2}\sqrt{m}$

$h^{(A)} = m^{-\frac{1}{3}}$

$h^{(B)} = 2^{-\frac{1}{3}} m^{-\frac{1}{3}}$

Order of accuracy $O(h) \Rightarrow$ error reduce by factor of

$\underline{2^{\frac{1}{3}} \approx 1.26}$

Q1: This was a popular question. In question (a)i, there was some confusion with thermal conductivity problems, and several students struggled with quickly formulating the expression of the stiffness matrix restricted to one node only. Question (a)ii was the less successful question: most student forgot to include the effect of gravity in the external force vector and struggled with the integration of the boundary force (numerical errors). Also, the definition of the force vector's dimension was rarely clear (should be either 6 (for the whole element) or 2 (only for node 3)).

Part (b) was in general more successful, almost all students expressed the isoparametric mapping correctly. The definition of the Jacobian was in general correct, but most students were not comprehensive when explaining how it is involved in the calculation of the stiffness matrix (change of variable in the integral + in the calculation of derivatives in matrix B).

Q3: This was the most popular question. Most students did well. The most difficult part was to clearly explain the reason why the shape function proposed in (a) is not suitable. Question (b)iii was generally well answered, with few students getting confused with the assembly process: they did not see the simplification and got lost in tedious calculations.

Q2: The first parts of the question (a to c) were generally well answered by the students, with sometimes confusion about the polynomial order of Hermitian shape functions (although these functions were given in the databook). In the calculation of error reduction factor, students often missed the fact that the question was about the error in the displacement derivative, not the displacement.

All students struggled with the last part of the question (e), most of them correctly split the problem (and could explain why this was appealing), but none succeeded in calculating the element matrix for the split problem.

Q4: This was the least popular question, although the students who picked that question did well overall. There were some confusions between memory and time increase in (b) and the semi-norm definition in question (d) was often incorrect. In part (a), most students forgot that each node has 3 degrees of freedom.