In the revised examination paper the marks for Question 3 have been slightly altered.

F. Cirak

EGT2
ENGINEERING TRIPOS PART IIA

Monday 5 May 2014 2 to 3.30

Module 3D7

FINITE ELEMENT METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 Data Sheet (3 pages)
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Consider the one-dimensional differential equation for hydrogen diffusion through a solid

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right),$$

where $c(x,t)$ is the concentration of hydrogen at position x and time t and D is the diffusion coefficient.

(a) Derive the weak form of the above equation for a domain spanning $0 \leq x \leq L$ with a Dirichlet boundary condition $c = c_0$ at $x = 0$ and a Neumann boundary condition $j_0 = -D\frac{\partial c}{\partial x}$ at $x = L$ for all time t. [20%]

(b) The diffusion of hydrogen is affected by the known stress σ in the domain and the stress modified diffusion equation reads

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right) - \overline{D}\frac{\partial}{\partial x}\left(c\frac{\partial \sigma}{\partial x}\right),$$

where $\overline{D}$ is a constant.

(i) Write the weak form for this modified diffusion equation with boundary conditions as in part (a). [20%]

(ii) The semi-discrete global finite element problem for this modified diffusion equation can be expressed as

$$\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f},$$

where $\boldsymbol{u}$ is the vector of the nodal hydrogen concentrations. For a linear element of length h calculate the elemental contributions to $\boldsymbol{M}$, $\boldsymbol{K}$ and $\boldsymbol{f}$ in terms of the given nodal stress values (σ_1, σ_2). You may assume that the diffusion coefficient D is a constant. [40%]

(iii) Qualitatively discuss a solution strategy for the semi-discrete global finite element problem in part (ii) when the diffusion coefficient D is a function of the concentration c. [20%]

2 (a) Figure 1(a) shows a finite element mesh consisting of one single three-noded triangular element. The element represents a planar elastic sheet with Young's modulus $E = 200$ and Poisson's ratio $\nu = 0$ under plane stress condition. In the following consider only the unconstrained degrees of freedom.

(i) Determine the shape functions of the element. [10%]

(ii) Determine the strain-displacement matrix $\boldsymbol{B}^e$. [20%]

(iii) Determine the stiffness matrix. [30%]

(b) Figure 1(b) shows a finite element mesh consisting of two three-noded triangular elements. The mesh discretises a planar elastic sheet with the same material parameters as in part (a) and is loaded as shown in Fig. 1(b) with a force f_x applied to the node connected to the inclined roller support.

(i) Determine the global stiffness matrix of the two elements. [10%]

(ii) Determine the displacement of the node connected to the roller support. [30%]

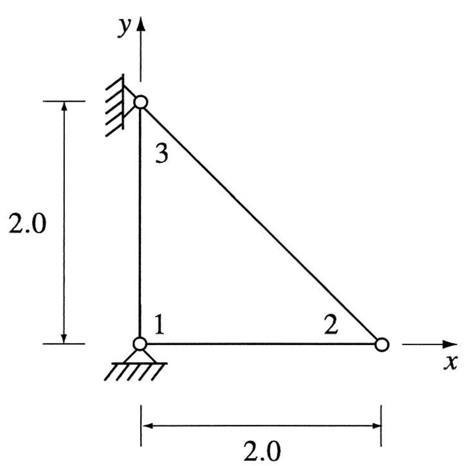


Fig. 1(a)

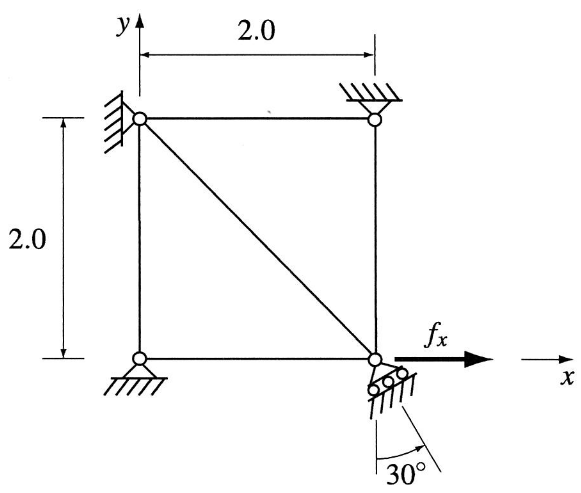


Fig. 1(b)

3 (a) Figure 2(a) shows the finite element mesh for a plate with a hole under axial tension of which only a quarter is discretised. Comment on the suitability of the shown mesh for finite element computations. Propose an alternative discretisation by sketching a finite element mesh. [10%]

(b) Figure 2(b) shows a four-noded isoparametric element.

(i) Compute the Jacobian matrix of the element. [35%]

(ii) The displacement vector of the element is given by

$$\boldsymbol{a}_e = \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} & u_{x3} & u_{y3} & u_{x4} & u_{y4} \end{bmatrix}^T$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0.2 \end{bmatrix}^T .$$

Compute the strain components ε_{xx} and ε_{yy}. [30%]

(c) Figure 2(c) shows a tetrahedral element with four nodes. Write down the equation system for determining the corresponding shape functions. [25%]

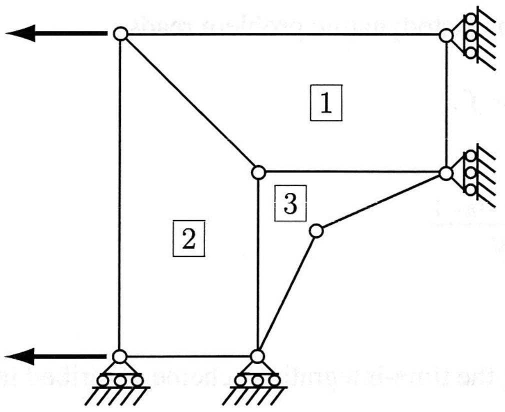


Fig. 2(a)

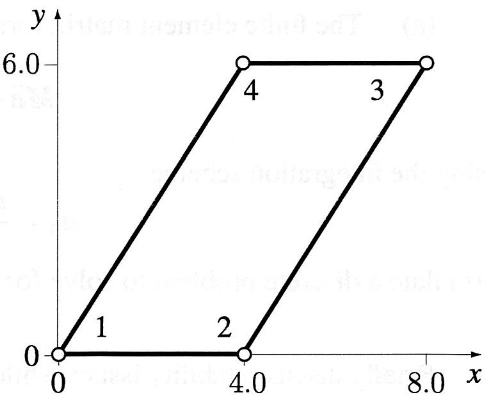


Fig. 2(b)

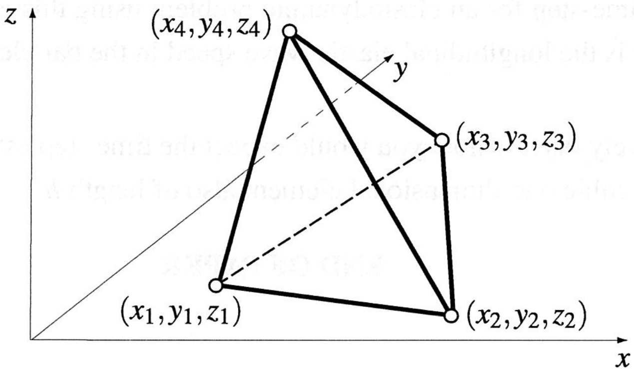


Fig. 2(c)

(TURN OVER

4 (a) The finite element matrix form of an elastodynamic problem reads

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}.$$

Using the integration scheme

$$\dot{\boldsymbol{a}}_n = \frac{\boldsymbol{a}_{n+1} - \boldsymbol{a}_{n-1}}{2\Delta t}$$

formulate a discrete problem to solve for $\boldsymbol{a}_{n+1}$. [30%]

(b) Briefly discuss stability issues while using the time-integration scheme described in part (a). [20%]

(c) Consider a one-dimensional linear bar element of length h with a constant stiffness EA and density ρ. By computing the natural frequency of this element show that the critical stable time-step for an elastodynamic problem using this element is proportional to h/c, where c is the longitudinal elastic wave speed in the bar element. [30%]

(d) Qualitatively discuss how you would expect the time step estimate made in part (c) to change for a cubic one-dimensional element also of length h. [20%]

END OF PAPER

3D7 DATA SHEET

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\, dV$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Heat flux	$\boldsymbol{q} = -\boldsymbol{D}\nabla T$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI\boldsymbol{B}\, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

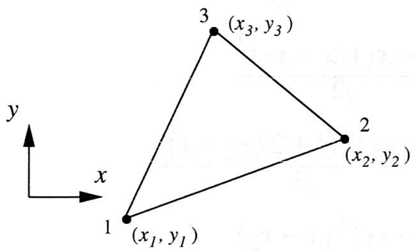
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D, isotropic)

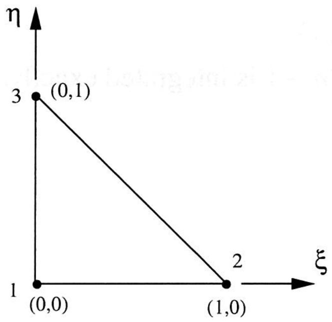
$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Shape functions

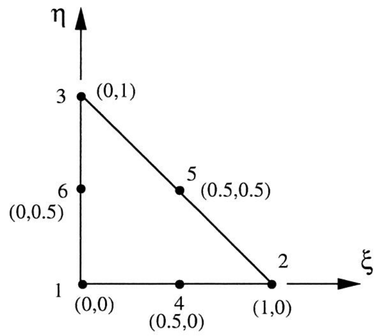


$$N_1 = ((x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y)/2A$$
$$N_2 = ((x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y)/2A$$
$$N_3 = ((x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y)/2A$$

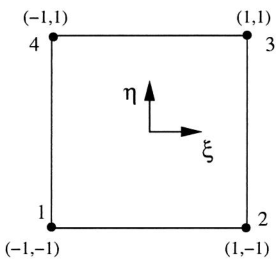
$A =$ area of triangle



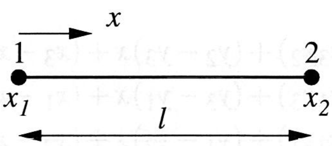
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)(1 - \eta)/4$$
$$N_2 = (1 + \xi)(1 - \eta)/4$$
$$N_3 = (1 + \xi)(1 + \eta)/4$$
$$N_4 = (1 - \xi)(1 + \eta)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1,1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$