EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 28 April 2021 1.30 to 3.10

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Attachment: 3D7 datasheet (3 pages).
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 A gravity dam with a cross-section of height $2L$ and width L is subject to hydrostatic pressure $p(y) = \rho_w g(2L - y)$, where ρ_w is the density of water and g is the acceleration due to gravity. The dam is made of a homogeneous and isotropic material of Young's modulus E, Poisson's ratio ν and density ρ_s.

(a) We model the dam using the 3 noded triangular element shown in Fig. 1(a). Nodes 1 and 2 have zero displacement.

(i) Compute the components of the stiffness matrix associated with node 3. [20%]

(ii) Compute the external force vector. [20%]

(iii) Is this mesh relevant to accurately predict the displacement at node 3? Why? [10%]

(b) We now model the dam using the 6 noded triangular element shown in Fig. 1(b), where the nodal coordinates of the new nodes are $(\frac{L}{2}, 0)$, (d, L) and $(0, L)$ for nodes 4, 5 and 6 respectively, d is a geometric parameter.

(i) Determine the isoparametric map for this element. [20%]

(ii) Compute the Jacobian of the element and explain how you would use it to compute the stiffness matrix of the element. [20%]

(iii) What happens to the determinant of the Jacobian when d varies within $0.25L < d < 0.75L$? [10%]

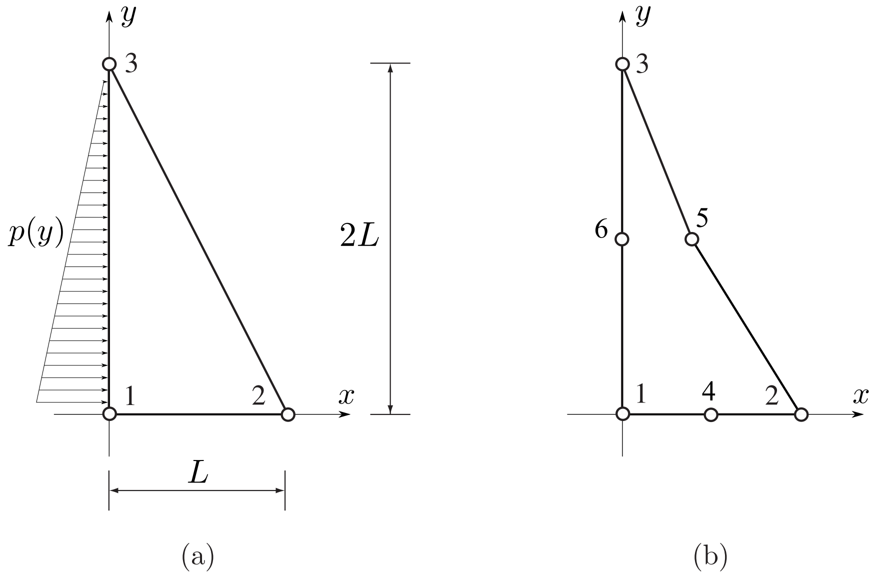


Fig. 1

2 Consider the equation:

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} = f \qquad \text{[equation 1]}$$

where f is given.

(a) Derive the weak formulation of [equation 1] for a problem where $u = 0$ and $\mathrm{d}^2u/\mathrm{d}x^2 = 0$ at the ends of the domain. [20%]

(b) What type of finite element shape function would you recommend for this problem? [10%]

(c) For the lowest order suitable shape functions for this problem, how many Gauss quadrature points would you recommend? Justify your answer. [10%]

(d) For the lowest order suitable shape functions for this problem, by what factor would you expect the error in $\mathrm{d}u/\mathrm{d}x$ to decrease if the number of elements for a problem is doubled? [10%]

(e) Split [equation 1] into two second-order differential equations, and compute the element matrix for the split problem using lowest order admissible shape functions. Comment on why splitting [equation 1] might be an appealing approach for this problem. [50%]

3 (a) For computing the shape functions of a three-noded triangular element we consider the function $T(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y$ with the α_i's being the coefficients to be determined. Explain why you should not alternatively use the function $T(x,y) = \alpha_0 + \alpha_1 x^2 + \alpha_2 y^2$. [15%]

(b) Figure 2(a) shows a square-shaped domain representing an isotropic material with a heat conductivity $k = 3$. Throughout the domain there is a uniform heat source with $s = 2$. Along the four boundary edges the temperature is prescribed to be zero. The domain is discretised with the triangular element shown in Fig. 2(b).

(i) Compute the element source vector $\vec{f}^e$ of the three-noded triangular element. [25%]

(ii) Compute the element conductance matrix $\vec{K}^e$ of the three-noded triangular element. [40%]

(iii) Using the computed element conductance matrix and the element source vector determine the temperature at the centre of the square domain. [20%]

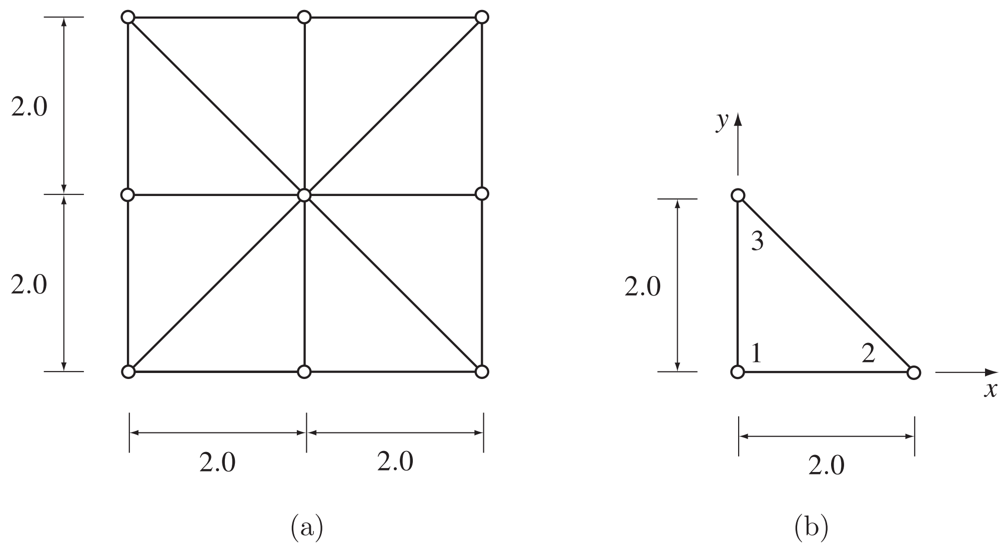


Fig. 2

4 Two finite element simulations of a static elasticity problem are performed using a structured mesh of trilinear hexahedral elements. Each element has eight nodes and forms a cube, and all elements in a given mesh have the same size. Case A uses m elements and Case B uses $2m$ elements, where m is large.

(a) How many non-zero entries would you expect in most rows of the global stiffness matrix for Case A and for Case B? [20%]

(b) Approximate the increase in time and memory required to build the global stiffness matrix for Case B compared to Case A. [20%]

(c) Cases A and B are solved using a sparse LU solver, and then again with a multigrid preconditioned solver. The cost complexities are $O(n^2)$ for the sparse LU solver and $O(n)$ for the multigrid solver. For each solver type estimate the factor increase in linear solver time in going from Case A to Case B. [20%]

(d) Given the exact displacement field u and the finite element solution u_h, express the error in the H^1 semi-norm. [20%]

(e) The error in the H^1 semi-norm is proportional to Ch, where h is the length of a typical element edge and C is a problem constant that does not depend on h. Estimate the factor reduction in the strain error for Case B compared to Case A. [20%]

END OF PAPER

Engineering Tripos Part IIA
Module 3D7: Finite Element Methods

Data Sheet

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\, dV$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI\boldsymbol{B}\, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

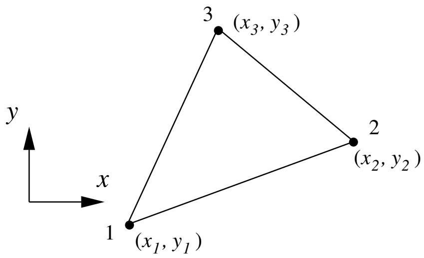
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D)

$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

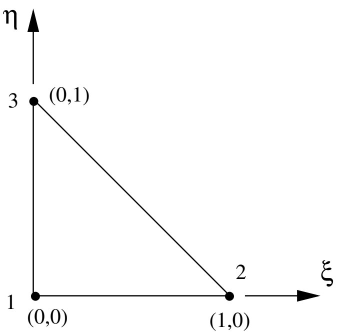
Shape functions



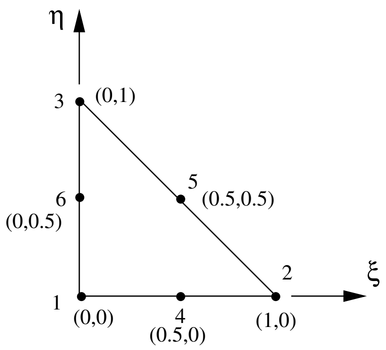
$$N_1 = \big(\left(x_2 y_3 - x_3 y_2\right) + \left(y_2 - y_3\right) x + (x_3 - x_2)\, y\big)/2A$$
$$N_2 = \big(\left(x_3 y_1 - x_1 y_3\right) + \left(y_3 - y_1\right) x + (x_1 - x_3)\, y\big)/2A$$
$$N_3 = \big(\left(x_1 y_2 - x_2 y_1\right) + \left(y_1 - y_2\right) x + (x_2 - x_1)\, y\big)/2A$$

$A =$ area of triangle

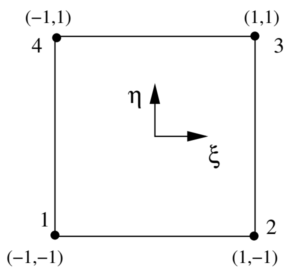
$\int_\Omega N_I = A/3,\ i = 1,\ 2,\ 3.$



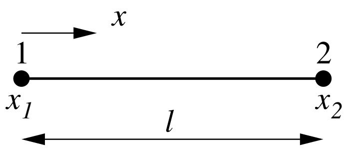
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\left(1 - \xi - \eta\right)^2 - \left(1 - \xi - \eta\right)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\left(1 - \xi - \eta\right)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\left(1 - \xi - \eta\right)$$



$$N_1 = (1 - \xi)\left(1 - \eta\right)/4$$
$$N_2 = (1 + \xi)\left(1 - \eta\right)/4$$
$$N_3 = (1 + \xi)\left(1 + \eta\right)/4$$
$$N_4 = (1 - \xi)\left(1 + \eta\right)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$
$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$
$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$
$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1, 1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$