Question 1

(a) Multiply by a test function: Let $v(x)$ be a test function satisfying $v(0) = v(1) = 0$. Multiply the equation by $v(x)$ and integrate:

$$\int_0^1 \left(\frac{d^4 w}{dx^4} + w \right) v\, dx = \int_0^1 v\, dx.$$

Apply integration by parts twice to the fourth-order term. Boundary terms vanish due to $v(0) = v(1) = 0$ and $w''(0) = w''(1) = 0$:

$$\int_0^1 \frac{d^4 w}{dx^4} v\, dx = \int_0^1 \frac{d^2 w}{dx^2} \frac{d^2 v}{dx^2}\, dx.$$

Substitute back to obtain:

$$\int_0^1 \frac{d^2 w}{dx^2} \frac{d^2 v}{dx^2}\, dx + \int_0^1 wv\, dx = \int_0^1 v\, dx \quad \forall v \in H_0^2(0,1).$$

(b) First we check the boundary conditions with the trial solution:

1. $w(0) = 0 \implies C_3 = 0.$
2. $w(1) = 0 \implies C_3 = 0.$
3. $\frac{d^2 w}{dx^2}(0) = 0$ and $\frac{d^2 w}{dx^2}(1) = 0$ are automatically satisfied by the trial solution.

We now take a test function $v(x) = \sin(\pi x)$:

1. Compute derivatives:
$$w''(x) = -\pi^2 C_1 \sin(\pi x) - 9\pi^2 C_2 \sin(3\pi x), \quad v''(x) = -\pi^2 \sin(\pi x).$$
2. First term in weak formulation:
$$\int_0^1 w'' v''\, dx = \pi^4 C_1 \int_0^1 \sin^2(\pi x) dx = \frac{\pi^4 C_1}{2}.$$
3. Second term in weak formulation:
$$\int_0^1 wv\, dx = C_1 \int_0^1 \sin^2(\pi x) dx = \frac{C_1}{2}.$$
4. Right-hand side:
$$\int_0^1 v\, dx = \frac{2}{\pi}.$$
5. Combine terms:
$$\frac{\pi^4 C_1}{2} + \frac{C_1}{2} = \frac{2}{\pi} \implies \boxed{C_1 = \frac{4}{\pi(\pi^4 + 1)}.}$$

We now take another test function $v(x) = \sin(3\pi x)$

1. Compute derivatives:
$$w''(x) = -\pi^2 C_1 \sin(\pi x) - 9\pi^2 C_2 \sin(3\pi x), \quad v''(x) = -9\pi^2 \sin(3\pi x).$$

2. First term in weak formulation:
$$\int_0^1 w''v''\,dx = 81\pi^4 C_2 \int_0^1 \sin^2(3\pi x)dx = \frac{81\pi^4 C_2}{2}.$$

3. Second term in weak formulation:
$$\int_0^1 wv\,dx = C_2 \int_0^1 \sin^2(3\pi x)dx = \frac{C_2}{2}.$$

4. Right-hand side:
$$\int_0^1 v\,dx = \frac{2}{3\pi}.$$

5. Combine terms:
$$\frac{81\pi^4 C_2}{2} + \frac{C_2}{2} = \frac{2}{3\pi} \implies \boxed{C_2 = \frac{4}{3\pi(81\pi^4+1)}.}$$

Therefore,
$$\boxed{C_1 = \frac{4}{\pi(\pi^4+1)}, \quad C_2 = \frac{4}{3\pi(81\pi^4+1)}, \quad C_3 = 0.}$$

Q2)

a) - Insufficient boundary conditions (possible rigid body deformations)

- Too few quadrature points

- Inverted elements

- Something wrong with material (e.g. zero Young's mod)

b) In FE equilibrium satisfied only weakly, in the weak form we use only a limited set of functions as test and trial functions.

c) Gaussian quadrature is very efficient for polynomials and almost polynomials. Needs far fewer evaluation points than other schemes.

d) i)

$$x = \sum_{i=1}^{4} N_i(\xi,\eta)\, x_i = N_2(\xi,\eta) + \frac{3}{4} N_3(\xi,\eta)$$

$$= \frac{1}{16}\left(7 - \eta + 7\xi - \eta\xi\right)$$

$$y = \frac{3}{4} N_3(\xi,\eta) + N_4(\xi,\eta)$$

$$= \frac{1}{16}\left(7 + 7\eta - \xi - \eta\xi\right)$$

ii) $$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 7-\eta & -1-\eta \\ -1-\xi & 7-\xi \end{pmatrix}$$

iii) $$x(0,0) = \frac{7}{16} \qquad y(0,0) = \frac{7}{16}$$

$$s(0,0) = e^{\frac{49}{256}}$$

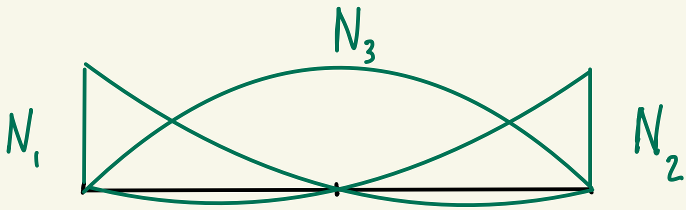
$$J(0,0) = \frac{1}{16} \begin{pmatrix} 7 & -1 \\ -1 & 7 \end{pmatrix} \qquad \det J(0,0) = \frac{3}{16}$$

$$f = 4 \cdot \frac{3}{16} e^{49/256} \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{3}{16} e^{49/256} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Obviously, using a single quadrature point gives only a very rough approx. Need more Gauss points, possibly 2x2.

Q3) a)

$$N_1(\xi) = 1/2\, \xi(\xi-1)$$
$$N_2(\xi) = 1/2\, \xi(\xi+1)$$
$$N_3(\xi) = -(\xi-1)(\xi+1)$$



b) $$N_3(\xi,\eta) = \frac{1}{4}\xi\eta(\xi+1)(\eta+1)$$

$$N_9(\xi,\eta) = (\xi-1)(\xi+1)(\eta-1)(\eta+1)$$

$$N_8(\xi,\eta) = -\frac{1}{2}(\eta-1)(\eta+1)\xi(\xi-1)$$

c) Curved boundaries and higher order convergence for smooth problems.

d) i) Weak form

$$\int (\underline{\alpha}\cdot\nabla u - \nabla\cdot(\beta\nabla u) + f)\, w \, d\Omega = 0$$

$$\beta \int \nabla \cdot \nabla u \, w \, d\Omega = \beta \int \nabla \cdot (\nabla u \, w) \, d\Omega - \beta \int \nabla u \nabla w \, d\Omega$$

$$\int \nabla \cdot (\nabla u \, w) \, d\Omega = \int_{\Gamma_N} w \nabla u \cdot \underline{n} \, d\Gamma + \cancel{\int_{\Gamma_D} w \nabla u \cdot n \, d\Gamma}$$

Choose $w = 0$ on Γ_D

$$\Rightarrow \int \left(\underline{\alpha} \cdot \nabla u w - \beta \nabla u \cdot \nabla w \right) d\Omega + \int f w \, d\Omega + \int_{\Gamma_N} w \nabla u \cdot \underline{n} \, d\Gamma = 0$$

ii)

$$u = \sum_{i=1}^{3} N_i(x,y) u_i \qquad w = \sum_{i=1}^{3} N_i(x,y) w_i$$

$$N_1(x,y) = 1 - x - y$$

$$N_2(x,y) = x$$

$$N_3(x,y) = y$$

$$\nabla u = \begin{pmatrix} -u_1 + u_2 \\ -u_1 + u_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \underline{B} \, \underline{u}$$

Stiffness corresponding to $\beta \nabla u \cdot \nabla w$

$$\frac{1}{2} \beta \underline{B}^T \underline{B} = \frac{\beta}{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Stiffness corresponding to $\underline{\alpha}\cdot\nabla v\cdot\nabla w$

$$\alpha\cdot\nabla v = \alpha_x(-v_1+v_2)+\alpha_y(-v_1+v_3)$$

$$\begin{pmatrix}\alpha_x-\alpha_y\\ \alpha_x\\ \alpha_y\end{pmatrix}\cdot\int\begin{pmatrix}1-x-y\\ x\\ y\end{pmatrix}^T dx\,dy$$

Note that

$$\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx=\frac{1}{6}\qquad\int_0^1\int_0^{1-x}x\,dy\,dx=\frac{1}{6}$$

$$\Rightarrow\quad\frac{1}{6}\begin{pmatrix}\alpha_x-\alpha_y & \alpha_x-\alpha_y & \alpha_x-\alpha_y\\ \alpha_x & \alpha_x & \alpha_x\\ \alpha_y & \alpha_y & \alpha_y\end{pmatrix}$$

Question 4

(a)
The semi-discrete finite element system is:

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}.$$

Using the average acceleration scheme:

$$\begin{aligned}
\boldsymbol{a}_{n+1} &= \boldsymbol{a}_n + \Delta t\,\dot{\boldsymbol{a}}_n + \frac{\Delta t^2}{4}\left(\ddot{\boldsymbol{a}}_n + \ddot{\boldsymbol{a}}_{n+1}\right), \\
\dot{\boldsymbol{a}}_{n+1} &= \dot{\boldsymbol{a}}_n + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{a}}_n + \ddot{\boldsymbol{a}}_{n+1}\right).
\end{aligned}$$

Substitute the equation of motion $\ddot{\boldsymbol{a}}_{n+1} = \boldsymbol{M}^{-1}(\boldsymbol{f}_{n+1} - \boldsymbol{K}\boldsymbol{a}_{n+1})$ into the displacement update:

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \Delta t\,\dot{\boldsymbol{a}}_n + \frac{\Delta t^2}{4}\left(\ddot{\boldsymbol{a}}_n + \boldsymbol{M}^{-1}(\boldsymbol{f}_{n+1} - \boldsymbol{K}\boldsymbol{a}_{n+1})\right).$$

Rearrange terms and multiply through by $\boldsymbol{M}$:

$$\left(\boldsymbol{M} + \frac{\Delta t^2}{4}\boldsymbol{K}\right)\boldsymbol{a}_{n+1} = \boldsymbol{M}\boldsymbol{a}_n + \Delta t\boldsymbol{M}\dot{\boldsymbol{a}}_n + \frac{\Delta t^2}{4}\boldsymbol{M}\ddot{\boldsymbol{a}}_n + \frac{\Delta t^2}{4}\boldsymbol{f}_{n+1}.$$

Substitute $\ddot{\boldsymbol{a}}_n = \boldsymbol{M}^{-1}(\boldsymbol{f}_n - \boldsymbol{K}\boldsymbol{a}_n)$:

$$\boxed{\left(\boldsymbol{M} + \frac{\Delta t^2}{4}\boldsymbol{K}\right)\boldsymbol{a}_{n+1} = \left(\boldsymbol{M} - \frac{\Delta t^2}{4}\boldsymbol{K}\right)\boldsymbol{a}_n + \Delta t\boldsymbol{M}\dot{\boldsymbol{a}}_n + \frac{\Delta t^2}{4}(\boldsymbol{f}_n + \boldsymbol{f}_{n+1}).}$$

(b)
This is an implicit method, as it requires solving a system with $\boldsymbol{M} + \frac{\Delta t^2}{4}\boldsymbol{K}$. It is unconditionally stable for linear systems, with second order accuracy ($O(\Delta t^2)$).

(c)
A lumped mass matrix is used because its inversion is trivial and hence can significantly enhance the efficiency in explicit/semi-explicit methods. Lumped mass matrix can be constructed by row-sum lumping: $M_{ii}^{\text{lumped}} = \sum_j M_{ij}$, from the consistent mass matrix M.