EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 27 April 2022 2 to 3.40

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each extra sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 datasheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Figure 1 shows a clamped elastic beam of length L, constant Young's modulus E and constant second moment of area I. Its deflection u is determined by the governing equation

$$EI\frac{d^4u}{dx^4} = f$$

where f is the distributed load.

(a) Give the boundary conditions of this problem. [10%]

(b) Derive a suitable weak form of the governing equation. [50%]

(c) For a finite element formulation of this problem, comment on the required continuity of the shape functions. Explain which alternative model can be used if the shape functions do not possess sufficient continuity. [20%]

(d) Describe how the moment m and shear force q are obtained from the finite element solution. Comparing the moment m and shear force q, which one is in general more accurate? Explain why. [20%]

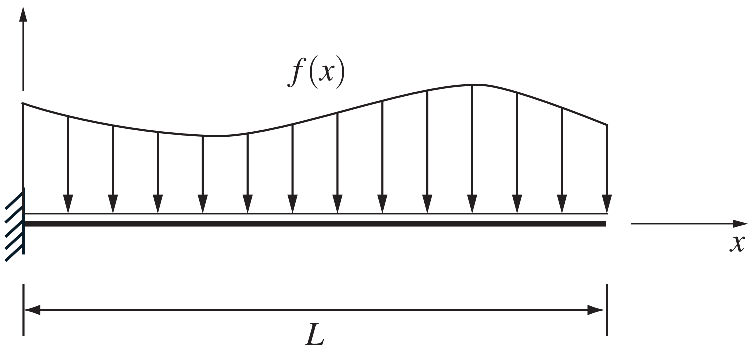


Fig. 1

2 On the domain Ω with boundary Γ, the following modified two-dimensional heat equation is given

$$\nabla^2 T + cT + s = 0$$

where T is the temperature, c is a constant and s is the heat source. The square of the del operator is defined as $\nabla^2 = \nabla \cdot \nabla$.

(a) Derive the weak form of the modified heat equation when the boundary condition over the entire boundary is $T = 0$. [30%]

(b) Derive the weak form of the modified heat equation when the boundary condition over the entire boundary is

$$\nabla T \cdot \boldsymbol{n} + \beta T = \overline{q}$$

where $\boldsymbol{n}$ is the unit normal to the boundary, β is a constant and $\overline{q}$ is the prescribed normal flux. [20%]

(c) Figure 2(a) shows a triangular domain discretised with two three-noded triangular elements as shown in Fig. 2(b). The boundary condition over the entire boundary is as in part (b). Compute the (2,2) and (2,3) components of the global stiffness matrix of the modified heat equation. [50%]

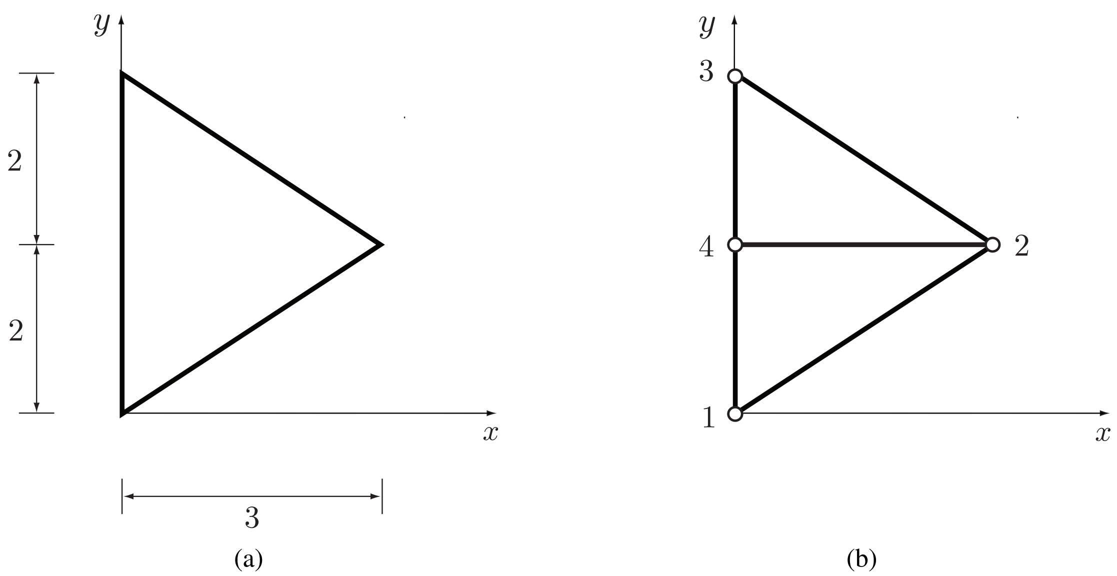


Fig. 2

3 (a) Explain why isoparametric mapping is important for deriving general two and three-dimensional finite elements. [5%]

(b) Figure 3(a) shows a four-noded isoparametric element.

(i) Compute the Jacobian matrix of the element. [35%]

(ii) The displacement vector of the element is given by

$$\begin{aligned}\boldsymbol{a}_e &= \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} & u_{x3} & u_{y3} & u_{x4} & u_{y4} \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.2 \end{bmatrix}^T\end{aligned}$$

Compute the strain components ϵ_{xy} and ϵ_{yy}. [30%]

(c) Figures 3(b) and 3(c) show two finite elements one of which has five nodes and the other has six nodes. Give an analytical expression for the shape functions corresponding to the following nodes and elements:

(i) node 1 of the five-noded triangle shown in Fig. 3(b); [15%]

(ii) node 5 of the six-noded quadrilateral element shown in Fig. 3(c). [15%]

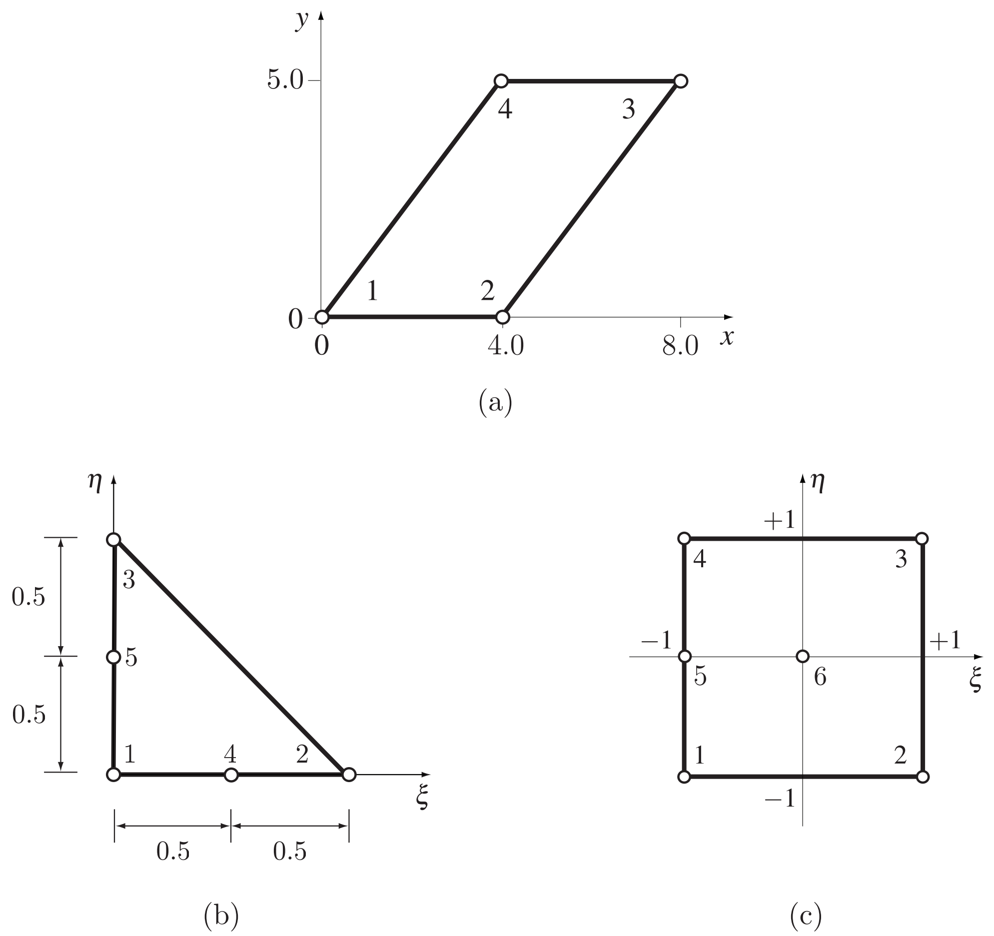


Fig. 3

4 Consider a one-dimensional bar of length L, density ρ, constant Young's modulus E and constant area A. The semi-discrete finite element equation of the bar discretised using a single two-noded element is given by

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$$

(a) Determine the stiffness matrix $\boldsymbol{K}$. [5%]

(b) Determine the consistent mass matrix $\boldsymbol{M}$. [15%]

(c) Determine the eigenfrequencies and eigenmodes of the bar. [40%]

(d) Determine the critical time step for the forward Euler method using your result from part (c). [40%]

END OF PAPER

3D7 DATA SHEET

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\,\mathrm{d}V$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T\boldsymbol{f}\,\mathrm{d}V$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Heat flux	$\boldsymbol{q} = -\boldsymbol{D}\nabla T$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\,\mathrm{d}V$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI\boldsymbol{B}\,\mathrm{d}V$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

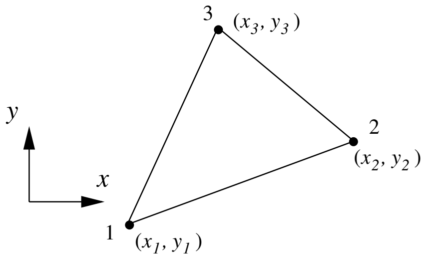
2D plane stress

$$\boldsymbol{D} = \frac{E}{\left(1-\nu^2\right)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D, isotropic)

$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

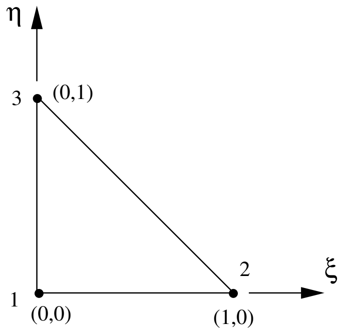
Shape functions



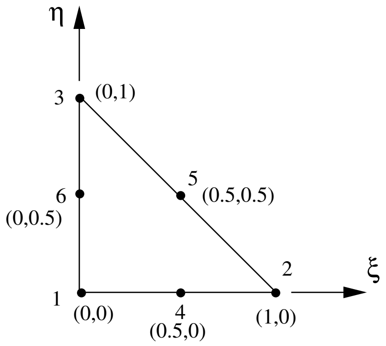
$$N_1 = ((x_2 y_3 - x_3 y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y)\,/2A$$
$$N_2 = ((x_3 y_1 - x_1 y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y)\,/2A$$
$$N_3 = ((x_1 y_2 - x_2 y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y)\,/2A$$

A = area of triangle

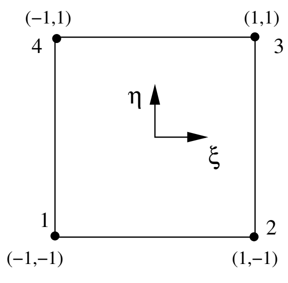
$$\int_\Omega N_i = A/3, \;\; i = 1,\ 2,\ 3.$$



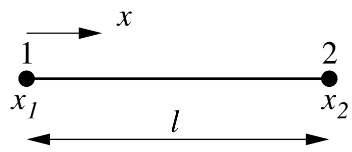
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\,(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\,(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\,(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)\,(1 - \eta)\,/4$$
$$N_2 = (1 + \xi)\,(1 - \eta)\,/4$$
$$N_3 = (1 + \xi)\,(1 + \eta)\,/4$$
$$N_4 = (1 - \xi)\,(1 + \eta)\,/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2\,(-l+2\,(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)\,(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2\,(l+2\,(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2\,(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1, 1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$