Cribs for the 3E3 Modelling Risk Exam Paper (2015-2016) *

There are two different approaches for solving this problem: One is to consider the transition probability matrix at the beginning of the week, and another is to consider the transition probability matrix at the end of the week.

The first approach: Consider the transition probability matrix at the beginning of the week, which is recommended in the question.

Q1(a)(i) The Markov Chain has six states: 0, 1, 2, 3, 4, 5. The transition probability matrix is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0.15 & 0 & 0.85 \\ 0 & 0 & 0 & 0.2 & 0.15 & 0.65 \\ 0 & 0 & 0 & 0.35 & 0.2 & 0.45 \end{pmatrix}.$$

Q1(a)(ii) The initial state distribution is $q^0 = (0, 0, 0, 0, 1.0, 0)$. The state distribution after two weeks is

$$q^0 \times P \times P = (0, 0, 0, 0.2, 0.15, 0.65) \times P = (0, 0, 0, 0.2875, 0.1525, 0.56).$$

Q1(a)(iii) The steady state distribution is the solution of the following equations:

$$(u_0\ u_1\ u_2\ u_3\ u_4\ u_5)P = (u_0\ u_1\ u_2\ u_3\ u_4\ u_5).$$

The solution is

$$(u_0\ u_1\ u_2\ u_3\ u_4\ u_5) = (0, 0, 0, 0.27439, 0.138211, 0.587398).$$

Q1(a)(iv) Let $K = 10$ be the unit fixed order cost, $h = 0.5$ be the unit holding cost and $d = 2.0$ be the unit penalty for shortages. Also let p_i be the probability when demand D is equal to i, where $i = 0, 1, 2, 3, 4$, and x be the system state: the inventory level at the beginning of the week. Clearly, the (s, S) policy means that $x = 3, 4, 5$.

For each combination of the system state x at the beginning of the week and the demand realization D during the week, we can calculate (1) inventory level at the end of the week, (2) whether or not an order is needed at the end of the week (an order is needed when the inventory level at the end of the week is less than 3) and when an order is needed it is equal to 1 and otherwise it is equal to zero, and (3) the number of demand shortage for

*RF Version 1

Holding Cost	Demand D	0	1	2	3	4
	Probability p	0.15	0.2	0.35	0.25	0.05
state x	5	5	4	3	2	1
state x	4	4	3	2	1	0
state x	3	3	2	1	0	0

Fixed Cost	Demand D	0	1	2	3	4
	Probability p	0.15	0.2	0.35	0.25	0.05
state x	5	0	0	0	1	1
state x	4	0	0	1	1	1
state x	3	0	1	1	1	1

Shortage Cost	Demand D	0	1	2	3	4
	Probability p	0.15	0.2	0.35	0.25	0.05
state x	5	0	0	0	0	0
state x	4	0	0	0	0	0
state x	3	0	0	0	0	1

Figure 1: The calculations on the inventory level, the ordering decision and the demand shortage.

the week. For example, when $x = 5$ and $D = 2$, we have (1) inventory level at the end of the week is 3, (2) an order is not needed at the end of the week, and (3) the number of demand shortage for the week is zero. These calculations are done Figure 1. Based on the numbers in Figure 1, we can calculate the expected cost as follows

$$\begin{aligned} & u_5(K(p_3 \times 1 + p_4 \times 1) + h(5p_0 + 4p_1 + 3p_2 + 2p_3 + 1p_4) + 0) \\ + \quad & u_4(K(p_2 \times 1 + p_3 \times 1 + p_4 \times 1) + h(4p_0 + 3p_1 + 2p_2 + 1p_3 + 0p_4) + 0) \\ + \quad & u_3(K(p_1 \times 1 + p_2 \times 1 + p_3 \times 1 + p_4 \times 1) + h(3p_0 + 2p_1 + 1p_2 + 0p_3 + 0p_4) + d \times 1 \times p_4), \end{aligned}$$

which is equal to £6.258689024.

The second approach: Consider the transition probability matrix at the end of the week.

Q1(a)(i) The Markov Chain has six states: 0, 1, 2, 3, 4, 5. The transition probability matrix is

$$P = \begin{pmatrix} 0 & 0.05 & 0.25 & 0.35 & 0.2 & 0.15 \\ 0 & 0.05 & 0.25 & 0.35 & 0.2 & 0.15 \\ 0 & 0.05 & 0.25 & 0.35 & 0.2 & 0.15 \\ 0.3 & 0.35 & 0.2 & 0.15 & 0 & 0 \\ 0.05 & 0.25 & 0.35 & 0.2 & 0.15 & 0 \\ 0 & 0.05 & 0.25 & 0.35 & 0.2 & 0.15 \end{pmatrix}.$$

Q1(a)(ii) The initial state distribution is $q^0 = (0, 0, 0, 0, 1.0, 0)$. The state distribution after two weeks is

$$q^0 \times P \times P = (0.05, 0.25, 0.35, 0.2, 0.15, 0) \times P = (0.0675, 0.14, 0.255, 0.2875, 0.1525, 0.0975).$$

Q1(a)(iii) The steady state distribution is the solution of the following equations:

$$(u_0\ u_1\ u_2\ u_3\ u_4\ u_5)P = (u_0\ u_1\ u_2\ u_3\ u_4\ u_5).$$

The solution is

$$(u_0\ u_1\ u_2\ u_3\ u_4\ u_5) = (0.089228, 0.159959, 0.250102, 0.27439, 0.138211382, 0.08811).$$

Q1(a)(iv) Let $K = 10$ is the unit fixed order cost, $h = 0.5$ be the unit holding cost and $d = 2.0$ be the unit penalty for shortages.

When state $n \leq 2$, a fixed cost for an order is $K = 10$; otherwise there is no fixed order cost. When state is n, the holding cost is equal to nh. When state $n \leq 2$, a new order will bring the inventory level to 5 and we do not anticipate any shortage and hence there is no shortage penalty; when state $n = 3$, no order is made and there is a shortage when demand is equal to 4 and hence the shortage cost is $0.05 \times (4 - 3) \times d = 0.05d$; when state $n \geq 4$, we do not anticipate any shortage and hence there is no shortage penalty because demand is less than or equal to the inventory level.

The total expected cost is the sum of the expected fixed cost plus the expected holding cost and the expected shortage cost:

$$\begin{aligned} & u_0(K + 0h + 0s) \\ & +u_1(K + 1h + 0s) \\ & +u_2(K + 2h + 0s) \\ & +u_3(0 + 3h + 0.05s) \\ & +u_4(0 + 4h + 0s) \\ & +u_5(0 + 5h + 0s) \\ = & \\ & u_0(10 + 0 \times 0.5 + 0 \times 2.0) \\ & +u_1(10 + 1 \times 0.5 + 0 \times 2.0) \\ & +u_2(10 + 2 \times 0.5 + 0 \times 2.0) \\ & +u_3(0 + 3 \times 0.5 + 0.05 \times 2.0) \\ & +u_4(0 + 4 \times 0.5 + 0 \times 2.0) \\ & +u_5(0 + 5 \times 0.5 + 0 \times 2.0) \\ = & \text{£}6.258689024. \end{aligned}$$

Q1(b)(i) n: stage/product index

x_n: action taken in stage n, which is the amount of product n

(s_n, v_n): state of the system in stage n where s_n represents the number minutes available in state n and v_n represents the demand capacity for product 2 in stage n

p_n is the unit profit for product n

$s_{n+1} = g_n(s_n, x_n)$: state transition

$P_n(s_n, x_n)$: system profit in stage n. We have

$$P_n(s_n, v_n, x_n) = p_n x_n.$$

$f_n^*(s_n, v_n)$: optimal reward-to-go in stage n when state is (s_n, v_n).

We have the following optimality equations:

$$f_1^*(s_1, v_1) = \max_{0 \le 2x_1 \le s_1} \{2x_1 + f_2^*(s_1 - 2x_1, v_1)\}$$

and

$$f_2^*(s_2, v_2) = \max_{0 \le x_2 \le s_2, 0 \le x_2 \le v_2} 5x_2.$$

Q1(b)(ii) We use the backward induction approach for solving the dynamic programming problem.

Stage 2. Recall that

$$f_2^*(s_2, v_2) = \max_{0 \le x_2 \le s_2, 0 \le x_2 \le v_2} 5x_2.$$

Thus the optimal solution is $x_2 = \min\{s_2, v_2\}$ and the optimal value is $5 \min\{s_2, v_2\}$.

Stage 1. Recall that

$$f_1^*(s_1, v_1) = \max_{0 \le 2x_1 \le s_1} \{2x_1 + f_2^*(s_1 - 2x_1, v_1)\}.$$

Note that $s_1 = 430$ and $v_1 = 230$, which gives $0 \le x_1 \le 215$, $s_1 - 2x_1 = 430 - 2x_1$ and $v_1 = 230$. The optimality condition becomes

$$f_1^*(s_1, v_1) = \max_{0 \le x_1 \le 215} \{2x_1 + 5 \min\{430 - 2x_1, 230\}\},$$

or

$$f_1^*(s_1, v_1) = \max_{0 \le x_1 \le 215} \{\min\{2150 - 8x_1, 1150 + 2x_1\}\}.$$

The optimal solution is obtained at $x_1 = 100$ at which two straight lines intersect, and the optimal value is 1350.

As a result, $s_2 = s_1 - 2x_1 = 430 - 200 = 230$ and $v_2 = v_1 - 0 = 230$, which shows that

$$x_2 = \min\{s_2, v_2\} = 230.$$

In summary, the optimal solution is

$$x_1 = 100, x_2 = 230,$$

and the optimal profit is

£1350.

Q1(c) In the continuous (s, S) policy, if the current inventory level z is greater than s we do not purchase additional units, and if z is less than or equal to s we order $S - z$ units. Here s is called the reorder point and S is called the order level. Both s and S are decision variables.

In the newsvendor problem, the current inventory level z is always assumed to be equal to zero. Thus there is no need to determine s and the optimal order level S in the (s, S) policy is the same as the optimal order quantity Q in the newsvendor problem. It turns out that finding the optimal order level S can be done by solving a newsvendor problem. We note that the assumptions for the (s, S) policy and those for the newsvendor problem are different. For example, a fixed ordering cost K is assumed in the (s, S) policy, but not in the newsvendor problem.

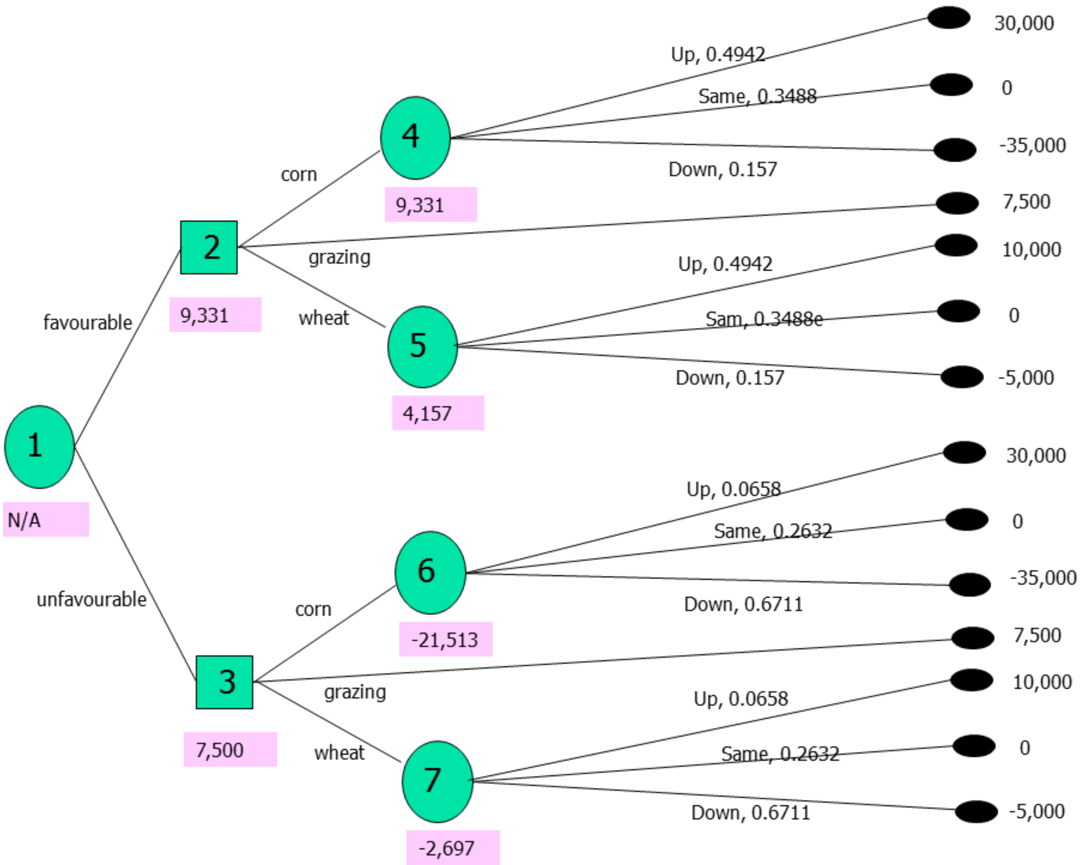

Figure 2: The decision tree.

Q2(a)(i) For the decision tree, see Figure 2.

Q2(a)(ii) We need to calculate the following conditional probabilities:

$$p(s_1|a_1), p(s_2|a_1), p(s_3|a_1), p(s_1|a_2), p(s_2|a_2), p(s_3|a_2).$$

By the Bayesian rule, we have

$$\begin{aligned} p(s_j|a_i) &= \frac{p(s_j \cap a_i)}{p(a_i)} \\ &= \frac{p(a_i|s_j)p(s_j)}{\sum_{m=1}^{3} p(a_i|s_m)p(s_m)}. \end{aligned}$$

and

$$p(s_1|a_1) = 0.49419, p(s_2|a_1) = 0.34884, p(s_3|a_1) = 0.15698,$$

$$p(s_1|a_2) = 0.06579, p(s_2|a_2) = 0.26316, p(s_3|a_2) = 0.67105.$$

We use the backward induction approach for finding the value of the project. See Figure 2. The optimal decision is the following: When the broker predicts favourable commodity prices, it is optimal to plant corns with the expected optimal value of £9,331; when the broker predicts unfavourable commodity prices, it is optimal to use the land as a grazing range with the expected optimal value of £7,500.

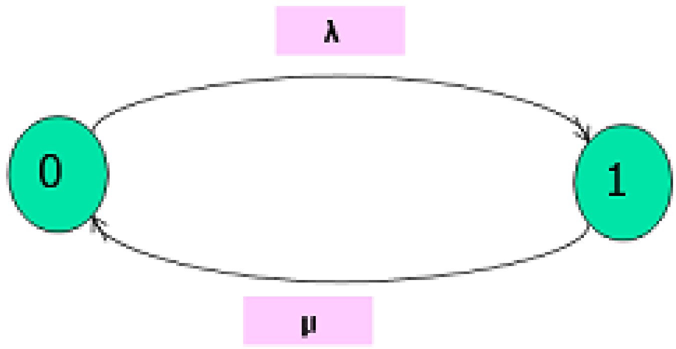

Figure 3: The queue diagram.

Q2(b)(i) For this $M/M/1$ queue with a capacity, there are two states 0 and 1. The arrival rate at state 0 is λ and the arrival rate at state 1 is 0 because customers never wait for service. The outgoing rate at state 1 is μ which is equal to the service rate and the outgoing rate at state 0 is 0 because there is no customer in the system. The queue diagram is shown in Figure 3. The balance equations at both states are the same and are the following

$$\lambda P_0 = \mu P_1,$$

where (P_0, P_1) are the steady-state probability distribution.

Q2(b(ii)) Note that $P_0 + P_1 = 1$. The above balance equation and the total probability equation give the solution:

$$P_0 = \frac{\mu}{\lambda + \mu}, \qquad P_1 = \frac{\lambda}{\lambda + \mu}.$$

Q2(b)(iii) The average number of customers in the system is given by

$$L = \sum_{i=0}^{1} i P_i = P_1.$$

Clearly

$$L_q = 0, \qquad W_q = 0,$$

because no customer is willing to wait for service. Furthermore, we have

$$W = W_q + \frac{1}{\mu} = \frac{1}{\mu}.$$

By Little's law we have that the effective arrival rate in the system is

$$\frac{L}{W} = \frac{\lambda\mu}{\lambda + \mu},$$

which is the same as

$$P_0\lambda + P_1 \times 0 = P_0\lambda.$$

Q2(c)(i) A class for a Markov Chain is a set of states which communicate with each other and which do not communicate with the states outside of this set. Clearly, all three states of this Markov Chain belong to the same class because state 1 can access state 3 (0.6), state 3 can access state 2 (0.5) and state 2 can access state 1 (0.3). One can also draw a transition network to answer the question.

Q2(c)(ii) The period of a state is equal to the greatest common divisor of n such that $P_{ii}(n) > 0$ (the n-step transition probability from state i to state i is positive). The class is a period property – all members in the same class have the same period. A sufficient condition for a state i to be aperiodic is that there exists N such that the Markov Chain can start at state i and return to state i in N steps and $N+1$ steps. The period for all members of this Markov Chain is equal to 1 because they are aperiodic – state 2 can reach state 2 in one step (0.3) and state 2 can reach state 2 in two steps (from state 2 to state 3 (0.4) and from state 3 to state 2 (0.5)).

Q2(c)(iii) The first passage time H_{ij} from state i to state j is the number of transitions until the process hits state j if it starts at state i. The expected first passage time $E(H_{ij})$ from state i to state j is the expected value of the first passage time from state i to state j. If u_i is the probability that the system in state i, then it holds that $u_i = \frac{1}{E(H_{ii})}$.

Q3(a)(i) Let D denote the one period random demand, with mean $\mu = E[D]$ and variance σ^2. Let c be the unit cost, $r > c$ the selling price, and $e = 0$ the salvage value. If Q units are ordered, then $\min(Q, D)$ units are sold and $(Q - D)^+ = \max(Q - D, 0)$ units are wasted. The profit is given by $r \min(Q, D) - cQ$. The objective for the newsvendor is to maximize the expected profit:

$$\pi(Q) = rE[\min(Q, D)] - cQ.$$

Q3(a)(ii) The objective function can be rewritten as

$$\begin{aligned}
\pi(Q) &= r \int_0^\infty \min(Q, D) dF(D) - cQ \\
&= r(\int_0^Q D dF(D) + \int_Q^\infty Q dF(D) - cQ \\
&= r(QF(Q) - \int_0^Q F(D) dD + Q \int_Q^\infty dF(D)) - cQ \\
&= r(QF(Q) - \int_0^Q F(D) dD + Q - QF(Q)) - cQ \\
&= r(Q - \int_0^Q F(D) dD) - cQ.
\end{aligned}$$

The first order derivative of the objective function is the following

$$\begin{aligned}
\frac{\partial \pi}{\partial Q} &= r \frac{\partial E[\min(Q, D)]}{\partial Q} - c \\
&= r(1 - F(Q)) - c.
\end{aligned}$$

The optimal solution Q^* satisfies the following optimality condition:

$$r(1 - F(Q)) = c,$$

for which a nice economic interpretation is that at the optimal order quantity the marginal revenue, $r(1 - F(Q))$, is equal to the marginal cost. By the stationary condition again, we obtain the optimal order quantity:

$$Q^* = F^{-1}\left(\frac{r - c}{r}\right),$$

where F^{-1} is the inverse function of the cumulative distribution function F for the demand D.

Q3(b)(i) The correlation coefficient r represents the strength of the linear relationship between two variables X and Y. The correlation coefficient is between -1 and 1. If the correlation coefficient is close to 1, it means that X and Y are highly positively correlated and it means that if one variable goes up, so does another variable. If the correlation coefficient is close to -1, it means that X and Y are highly negatively correlated and it means that if one variable goes up, another variable goes down. If the correlation coefficient is close to 0, it means that X and Y are not or weakly correlated.

$$r = \frac{\text{Covar}(X, Y)}{S_X S_Y},$$

where $\text{Covar}(X, Y)$ is the covariance between X and Y, S_X and S_Y are the standard deviations of X and Y respectively.

The R-square statistic represents the proportion of the variation in dependent variable Y that can be explained by the linear regression equation (explaining how well the regression line fits the data). The R-square statistic is between 0 and 1, and the regression line fits the data well if it is close to 1.

$$R^2 = \frac{RSS}{TSS}.$$

For a simple regression,

$$RSS = \sum_n (a + bx_i - \bar{y})^2, \qquad RSS = \sum_n (y_i - \bar{y})^2.$$

The slope for an independent variable in a multiple regression equation is the coefficient for that independent variable in the regression equation and it represents the change of the dependent variable with a unit change of this independent variable with all other independent variables held unchanged.

Q3(b)(ii) For simple regression, it holds that $r \times r = R^2$ and $b = r\frac{S_Y}{S_X}$, where b is the slope of the regression equation, S_X and S_Y are the standard deviations of X and Y respectively.

Q3(b)(iii) We look at the following attributes in Summary Outputs when we assess the strength and weakness of a multiple regression model.

- R-square statistics. A larger R-square statistic implies that the regression model fits the data better.
- The standard error for the regression model. A smaller standard error gives a higher prediction power.
- The t-statistic (or p-value or confidence intervals) for the slope of each independent variable. In order for an independent variable to be significant in the regression model, a larger value of the t-statistic is preferred.
- We need to pay attention to multi-collinearity, which states that two independent variables are highly correlated and may give an incorrect impression that either of the two independent variables are not true drivers for the dependent variable.
- Other possible factors are: the error plot, the sign of the slope, sample size, other key drivers for the dependent variable, etc.

Q3(c) For Winter's additive exponential smoothing method, the three smoothing equations are the base:

$$E_t = \alpha(X_t - S_{t-c}) + (1-\alpha)(E_{t-1} + T_{t-1}),$$

the trend:

$$T_t = \beta(E_t - E_{t-1}) + (1-\beta)T_{t-1},$$

and the seasonality:

$$S_t = \gamma(X_t - E_t) + (1-\gamma)S_{t-c},$$

and the forecasting equation is

$$F_{t+k} = E_t + kT_t + S_{t+k-c}.$$

The main difference between Winter's additive exponential smoothing method and Winter's multiplicative exponential smoothing method is how the seasonality factors are taken into account in the three smoothing equations and the forecasting equation: it is additive in the former and multiplicative in the latter. When the seasonality factors are not taken into account, then all seasonality factors become zero in Winter's additive exponential smoothing method and one in Winter's multiplicative exponential smoothing method. In this case, both methods reduce to the exponential smoothing method with trend.

Q3(d) A portfolio is efficient if no other portfolio is better than this portfolio in both return and risk. The market portfolio is the efficient portfolio with the highest Sharpe ratio. Thus some people thinks that the market portfolio is a super efficient portfolio among all efficient portfolios. The Sharpe ratio for a portfolio is defined by the following slope

$$\frac{r - r_f}{\sigma - 0} = \frac{r - r_f}{\sigma},$$

if we assume that the return for the risk-free asset is r_f, the return and the risk for a portfolio are r and σ respectively. Note that one uses other risk measures such as the variance.

Q3(e) The capital market line relates the expected rate of return of an efficient portfolio to its standard deviation, but it does not show the expected rate of return of an individual asset relates to its individual risk.

If the market portfolio M is efficient, then the expected rate of return r_i of any asset i satisfies the following CAPM equation:

$$r_i - r_f = \beta_i (r_M - r_f),$$

where r_f, r_M and r_i are the returns for the risk-free asset, the market portfolio, and the asset i, and β_i is referred to as the beta of asset i. Thus CAPM states that the expected excess rate of return of an asset is proportional to the expected excess rate of return of the market portfolio with proportionality factor of beta.