1 a) Let $\delta_k = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases}$.

The <u>pulse response</u>, g_k, is the output of G with δ_k as input.

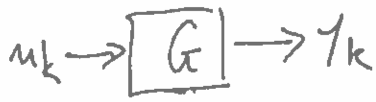
b) G needs to be <u>time invariant</u>

Then the output, $y_k = g_k * u_k$ (convolution)

$$= \sum_{i=0}^{k} g_i u_{k-i} = \sum_{i=0}^{k} g_{k-i} u_i$$

c) For G to be <u>stable</u>, any bounded input must give a bounded output. i.e.: for some $M, N > 0$

$|u_k| < M \;\; \forall k \Rightarrow |y_k| < N \;\; \forall k$.



Proof of proposition:

Suppose $\sum_{k=0}^{\infty} |g_k| = M_1 < \infty$

Let u_k be input such that $|u_k| < M_2 \;\; \forall k$

Then |output|, $|y_k| = \left| \sum_{i=0}^{k} g_i u_{k-i} \right|$

$$\leq \sum_{i=0}^{k} |g_i u_{k-i}| \quad \text{by } \underline{\text{triangle inequality}}$$

$$= \sum_{i=0}^{k} |g_i| |u_{k-i}|$$

$$< M_2 \sum_{i=0}^{k} |g_i| \leq M_1 M_2 \quad \therefore G \text{ stable.}$$

d)(i) $y' = ay + u$. Sample $y(t)$ as $y_k = y(\tau k)$

$y' \approx \frac{y_{k+1} - y_k}{\tau}$ $\therefore$ approx discrete-time soln. given by

$y_{k+1} - y_k = \tau(a y_k + u_k)$

$y_{k+1} = (1+\tau a) y_k + \tau u_k$ so $k(y_k) = (1+\tau a) y_k$

(ii) Take z transform:

$z\bar{y} - z y_0 = (1+\tau a)\bar{y} + \tau \bar{u}$, let $p = 1+\tau a$.

$(z-p)\bar{y} = z y_0 + \tau \bar{u}$ set $y_0 = 0$ for s.s. response.

Then T.F. $\frac{\bar{y}}{\bar{u}} = \frac{\tau}{z-p}$

Stable $\Leftrightarrow |p| < 1 \Leftrightarrow |1+\tau a| < 1 \Leftrightarrow -1 < 1+\tau a < 1$

$-2 < \tau a < 0$, $-2/a > \tau > 0$.

(iii) Solution will oscillate if $p < 0 \Rightarrow \tau > -1/a$

(iv) FIR $\Leftrightarrow p = 0 \Leftrightarrow \tau = -1/a$

Question 2.

(a.i) By backward difference, computing $P(s)$ for $s = \frac{z-1}{zT}$ gives

$$P\left(\frac{z-1}{zT}\right) = \frac{1}{\frac{z-1}{zT}} = \frac{zT}{z-1} = \frac{T}{1-z^{-1}} = \frac{1}{1-z^{-1}} \,.$$

Because of the sampling period $T = 1$, the delay can be represented directly with the delay operator $z^{-\tau}$, which results into the expression $\frac{z^{-\tau}}{1-z^{-1}}$ for $G(z)$.

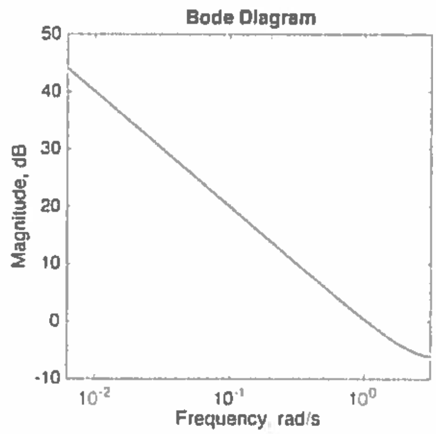
(a.ii) Method 1: $G(z)$ has a pole in 1 therefore is an infinite impulse response filter (finite impulse response filters have all poles in zero). Numerator and denumerator have the same degree, therefore the filter is causal.

Method 2: the difference equation of the filter reads $y(k) - y(k-1) = u(k-\tau)$. The filter is thus IIR since it is recursive. The actual value of the filter $y(k)$ does not depend on future values of the input $u(k+n)$, $n > 0$, therefore the filter is causal.

(b.i) For $\tau = 0$,

$$|G(e^{j\theta})|^2 = G(e^{j\theta})G(e^{j\theta})^* = \frac{1}{1-e^{-j\theta}} \cdot \frac{1}{1-e^{j\theta}} = \frac{1}{2-e^{-j\theta}-e^{j\theta}} = \frac{1}{2-2\cos(\theta)}$$

For $0 \le \theta \le \pi$, $|G(e^{j\theta})| = \sqrt{\frac{1}{2-2\cos(\theta)}}$ goes to infinity for $\theta \to 0$. $|G(e^{j\theta})|$ monotonically decreases as θ increases, reaching its minimum at $|G(\pi)| = \frac{1}{2}$. Finally, recall that $G(z)$ is the discretization of an integrator, therefore it is expected a decay of -20 dB/dec at low frequencies. The magnitude plot does not change for $\tau > 0$.



(b.ii) The step responses is unbounded since the magnitude of the Bode diagram goes to infinity when the frequency approaches 0.

Equivalently, in the z-domain the response of the filter to the step input $U(z) = \frac{1}{1-z^{-1}}$ reads

$$Y(z) = G(z)U(z) = \frac{z^{-\tau}}{(1-z^{-1})^2} = z^{-\tau}\left(\frac{1}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2}\right)$$

whose antitrasform reads

$$y(k) = \begin{cases} 1 + k - \tau & k \geq \tau \\ 0 & \text{otherwise.} \end{cases}$$

(b.iii) 20 dB corresponds to an amplification factor of 10. We look for θ at which $|G(e^{j\theta})| = 10$, that is, $|G(e^{j\theta})|^2 = \frac{1}{2-2\cos(\theta)} = 100$. Thus, $\theta = \arccos(1 - 1/200) \simeq 0.1$.

(c.i) Plot **A** corresponds to $\tau = 0$.

For $\tau = 0$ the transfer function reads $G(z) = \frac{z}{z-1}$. Thus, $G(e^{j\pi}) = G(-1) = \frac{-1}{-2} = 0.5$, which excludes **C** and **D**. Because of the pole in -1, the transfer function goes to infinity near 0^+ with $-\pi/2$ phase, but numerator and denominator have the same degree, which exclude the phase variation of **B**.

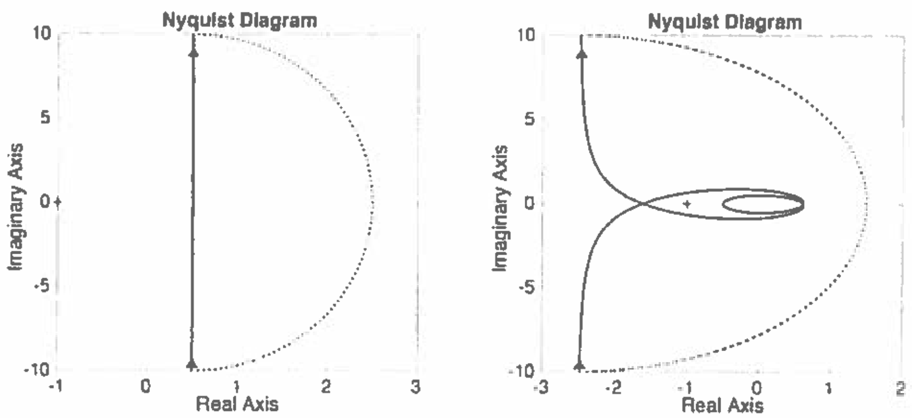
Equivalently, by explicit computation the real part of $G(e^{j\theta})$ is fixed at $\frac{1}{2}$ since

$$\frac{e^{j\theta}}{e^{j\theta}-1} = \frac{e^{j\theta}(e^{-j\theta}-1)}{(e^{j\theta}-1)(e^{-j\theta}-1)} = \frac{1-e^{j\theta}}{2-2\cos(\theta)} = \frac{1-\cos(\theta)-j\sin(\theta)}{2(1-\cos(\theta))} = \frac{1}{2} - \frac{j\sin(\theta)}{2(1-\cos(\theta))}.$$

Plot **C** corresponds to $\tau = 3$.

For $\tau = 3$ the transfer function reads $G(z) = \frac{1}{z^2(z-1)}$. Thus, $G(e^{j\pi}) = G(-1) = -0.5$, which excludes **A** and **B**. The two poles in zero gives a phase contribution of 2π for θ moving from 0^+ to π. This excludes **D**.

The complete Nyquist diagrams are provided below The dotted semicircle represents the closure at infinity.



(c.ii) For $\tau = 3$, closed loop stability holds for $K < \frac{1}{1.6}$. The open loop has no unstable poles therefore, by Nyquist criterion, we search the gain K that guarantees 0 encirclements of the point $-1/K$. For positive K, 0 encirclements are achieved if

$$-1/K < -1.6 \Rightarrow 1/K > 1.6 \Rightarrow K < \frac{1}{1.6}.$$

For $\tau = 0$, any positive gain K guarantees closed loop stability.

Question 3.

(a.i) Taking into account the frequency warping of the bilinear transform, we set The normalized cutoff frequencies are given by

$$\omega_l = \tan(0.25\pi/2) \simeq 0.4142 \qquad \omega_u = \tan(0.75\pi/2) \simeq 2.4142 \; .$$

Thus $\omega_l\omega_u = 1$ and $\omega_u - \omega_l = 2$ From the analog prototype $H(s) = \frac{1}{s+1}$, using the lowpass to bandpass analogue transform $s \to \frac{s^2+\omega_l\omega_u}{s(\omega_u-\omega_l)} = \frac{s^2+1}{2s}$ we get

$$H'(s) = \frac{2s}{s^2+2s+1} \; .$$

Using the bilinear transform we finally get

$$G(z) = H'(s)\big|_{s=\frac{z-1}{z+1}} = \frac{1}{2}\frac{z^2-1}{z^2}$$

(a.ii) The impulse response is obtained by antitransform

$$\begin{aligned} h_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\theta}\right)e^{j\theta n}d\theta = \frac{1}{2\pi}\left(\int_{-0.75\pi}^{-0.25\pi} e^{j\theta n}d\theta + \int_{0.25\pi}^{0.75\pi} e^{j\theta n}d\theta\right) \\ &= \frac{1}{2\pi}\left(\left[\frac{e^{j\theta k}}{jk}\right]_{-0.75\pi}^{-0.25\pi} + \left[\frac{e^{j\theta k}}{jk}\right]_{0.25\pi}^{0.75\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{e^{-j0.25\pi k}}{jk} - \frac{e^{-j0.75\pi k}}{jk} + \frac{e^{j0.75\pi k}}{jk} - \frac{e^{j0.25\pi k}}{jk}\right) \\ &= \frac{1}{\pi k}\left(\sin\left(0.75\pi k\right) - \sin\left(0.25\pi k\right)\right). \end{aligned}$$

At $k = 0$ we take the limit value $h_0 = 0.5$.

The use of a rectangular window of 11 samples gives

$$h'_k = \begin{cases} h_k & -5 \le k \le 5 \\ 0 & \text{otherwise} \end{cases}$$

The impulse response g_k of the filter and its transfer function $G(z)$ read

$$g_k = h'_{k-5} \qquad\qquad G(z) = \sum_{k=0}^{10} h'_{k-5}z^{-k} \; .$$

A wider window would reduce the transition band of the filter. Sharp discontinuity of the rectangular window results in side-lobe interference independent of the filter's order.

(b.i) Let g_k be the impulse response of the filter and let x_k be the input to the filter.

For a FIR filter of 11 samples and a 100-point DFT hardware, define $M = 10$ and $N = 100$. Standard convolution read

$$y_m = \sum_{k=0}^{\infty} g_k x_{m-k} = \sum_{k=0}^{M} g_k x_{m-k} \; .$$

Circular convolution read

$$y_m = \sum_{k=0}^{N-1} g_k x_{mod(m-k,N)}$$

where $mod(m-k,N)$ denotes $m-k$ in modulo N arithmetic. The two convolutions compute the same output for $M \leq m < n$.

(b.ii) FFT hardware can be used to compute the output of the filter, by taking advantage of the relation between circular and standard convolution.

In what follows we use vector notation for simplicity and we denote by FFT and IFFT respectively the fast Fourier transform and the inverse fast Fourier transform operations. We also use $M = 10$ and $N = 100$

- Processing via FFT hardware an input x_k of 200 samples requires frames

$$\begin{aligned} x_I &= [\underbrace{0,\ldots,0}_{M}, x_0, \ldots, x_{89}] \\ x_{II} &= [x_{80}, \ldots, x_{179}] \\ x_{III} &= [x_{170}, \ldots, x_{200}, 0, \ldots, 0] \end{aligned}$$

- Given the impulse response of the filter in vector notation

$$g = [g_0, \ldots, g_M, 0, \ldots, 0]$$

compute the FFT of the filter $G = \text{FFT}(g)$ and of each frame

$$X_I = \text{FFT}(x_I),\ X_{II} = \text{FFT}(x_{II}),\ X_{III} = \text{FFT}(x_{III}).$$

- Build the output frames by products

$$Y_I = GX_I,\ Y_{II} = GX_{II},\ Y_{III} = GX_{III}.$$

(standard product between vectors, G row vector, X_i column vector).

- Compute the inverse FFT

$$y_I = \text{iFFT}(Y_I),\ Y_{II} = \text{iFFT}(Y_I I),\ Y_{III} = \text{iFFT}(Y_I II).$$

- The output response of the filter is thus given by the last $N - M$ samples of y_I, y_{II} and y_{III} combined together.

(b.iii) The discrete Fourier transform applied to the impulse response of the filter read

$$\tilde{g}_m := \sum_{k=0}^{N-1} g_k e^{-j\frac{2\pi}{N}mk} \qquad 0 \leq m \leq N-1 .$$

Thus,

$$\tilde{g}_0 := \sum_{k=0}^{N-1} g_k$$

where g_k is given in part (a.ii).

4 a) WSS process $\{X(t)\}$ • $E[X(t)] = \mu$, const.

• $r_{xx}(t_1, t_2) = r_{xx}(t_2 - t_1) = r_{xx}(\tau)$

b) $r_{xx}(t_1, t_2) = E(X(t_1)X(t_2)) = E(X(t)X(t-\tau))$

$S_X = F.T.(r_{xx}(\tau))$

c) (i) $r_{xy}(t_1, t_2) = E(X(t_1)Y(t_2)) = E(X(t_1)\frac{d}{dt}X(t_2)) = \frac{d}{dt_2}E(X(t_1)X(t_2))$

$= \frac{d}{dt_2} r_{xx}(t_1, t_2)$

(ii) $r_{yy}(t_1, t_2) = E(X'(t_1)X'(t_2)) = \frac{d}{dt_1}E(X(t_1)X'(t_2)) = \frac{d}{dt_1} r_{xy}(t_1, t_2)$

iii) $r_{yy}(t_1, t_2) = \frac{d}{dt_1} r_{xy}(t_1, t_2) = \frac{d}{dt_1}\cdot\frac{d}{dt_2} r_{xx}(t_1, t_2)$; put $\tau = t_1 - t_2$

so that $\frac{\partial}{\partial t_2} r_{xx}(t_1, t_2) = -\frac{\partial}{\partial \tau} r_{xx}(\tau)$ and $\frac{\partial}{\partial t_1} r_{xx}(t_1, t_2) = \frac{\partial}{\partial \tau} r_{xx}(\tau)$

$\Longrightarrow r_{yy}(\tau) = -\frac{\partial^2}{\partial \tau^2} r_{xx}(\tau)$. (i)

Also, $E(Y(t)) = E(\frac{d}{dt}X(t)) = \frac{d}{dt}E(X(t)) = \frac{d}{dt}\mu = 0$ (ii)

(i) & (ii) $\Rightarrow$ Y is WSS.

iv) $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$. $\frac{d}{dt} \leftrightarrow i\omega$, $\therefore S_Y(\omega) = \omega^2 S_X(\omega)$

(or use part iii)

v) $Y_\tau(t) = \frac{X(t+\tau) - X(t-\tau)}{2\tau}$

Dirac delta fn.

impulse response $h(t) = \frac{1}{2\tau}(\delta(t+\tau) - \delta(t-\tau))$

$\therefore H(\omega) = F.T.(h(t)) = \int_{-\infty}^{\infty} e^{i\omega t}\frac{1}{2\tau}(\delta(t+\tau) - \delta(t-\tau))\,dt = \frac{e^{i\omega\tau} - e^{-i\omega\tau}}{2\tau}$

$= \frac{i\sin(\omega\tau)}{\tau} \rightarrow i\omega$ as $\tau \rightarrow 0$. Limit is T.F. of derivative operator.