EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 24 April 2018   9.30 to 11.10

Module 3F1

SIGNALS & SYSTEMS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Let G be a discrete time linear system.

(a) Define the *pulse response*, $\{g_k\}_{k \geq 0}$, of G. [5%]

(b) Suppose $\{u_k\}$ is an arbitrary input to G. Write down the output of G in terms of the pulse response and state any additional conditions G must satisfy for the expression to be valid. [15%]

(c) What does it mean for G to be *stable*? State and prove a sufficient condition on the quantity

$$\sum_{k=0}^{\infty} |g_k|$$

for G to be stable. [30%]

(d) Consider the (continuous time) ordinary differential equation

$$\frac{dy(t)}{dt} = ay(t) + u(t) \qquad (1)$$

with initial condition $y(0) = y_0$, where $a < 0$ is a constant.

(i) Derive a discrete time system of the form

$$y_{k+1} = h(y_k) + u_k$$

that provides an approximate solution to (1) with a sampling interval τ. [10%]

(ii) Using the z-transform, find conditions on τ that guarantee the stability of the discrete-time system. [20%]

(iii) For what values of τ will the solution oscillate? [10%]

(iv) Are there values of τ for which the discrete time system will have a finite impulse response? [10%]

2 Consider the filter

$$G(z) = \frac{z^{-\tau}}{1 - z^{-1}}$$

where τ is a non-negative integer.

(a) Discretization and properties.

(i) Using backward difference with sampling period $T = 1$ seconds, show that the transfer function $G(z)$ is the discretization of the continuous system $P(s) = \frac{1}{s}$ (integrator), with an additional delay τ. [10%]

(ii) Is $G(z)$ a finite impulse response filter or an infinite impulse response filter? Is the filter causal? Explain your answers. [10%]

(b) Bode diagram and steady-state response.

(i) Sketch the magnitude plot of the Bode diagram of $G(z)$ for $\tau = 0$ seconds (hint: compute $|G(e^{j\theta})|$). How would this plot change for $\tau > 0$? [20%]

(ii) Is the step response of $G(z)$ bounded or unbounded? Explain your answer. [10%]

(iii) Compute the frequency at which the filter gives an amplification of 20 dB. [10%]

(c) Nyquist criterion.

(i) The plots of $G(e^{j\theta})$ for θ ranging from 0^+ to π and for delays $0 \le \tau \le 3$ are shown in Fig. 2. Determine which plots correspond to $\tau = 0$ and to $\tau = 3$ (explain your answer) and sketch the complete Nyquist diagrams. [20%]

(ii) For $\tau = 3$, use the Nyquist criterion to determine the positive values of gain K for which the closed loop in Fig. 1 is stable. How would this range of values change for $\tau = 0$? [20%]

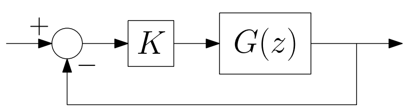

Fig. 1

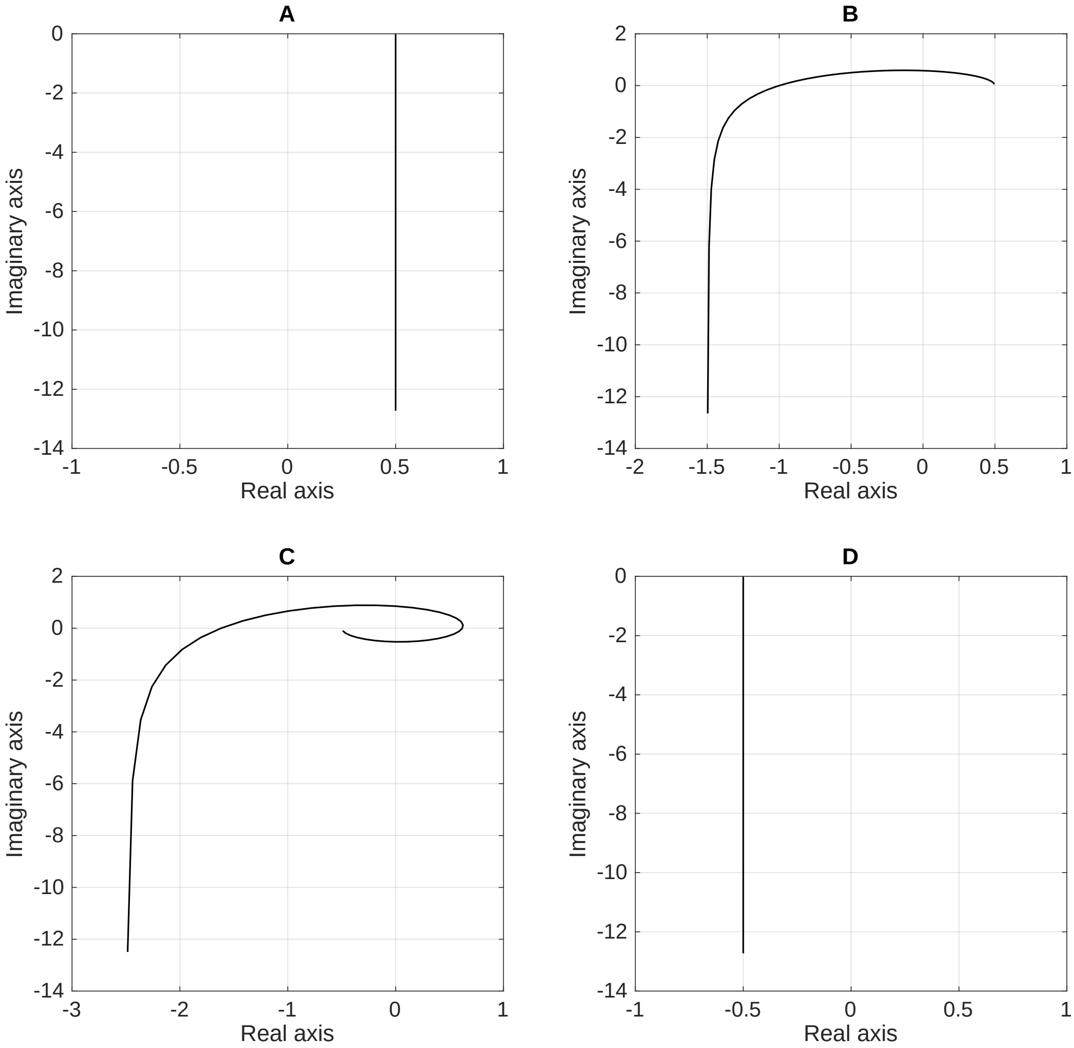

Fig. 2

3 A band pass filter $G(z)$ is required with 3dB normalized corner frequencies at 0.25π and 0.75π.

(a) IIR and FIR filter design.

(i) Design $G(z)$ as an infinite impulse response filter from the lowpass analogue prototype $\frac{1}{s+1}$, using the bilinear transform $s = \dfrac{z-1}{z+1}$ together with the lowpass to bandpass transformation $s = \frac{\bar{s}^2+\omega_l\omega_u}{\bar{s}(\omega_u-\omega_l)}$ (lower cutoff at ω_l and upper cutoff at ω_u). [25%]

(ii) Design $G(z)$ as a finite impulse response filter from the ideal bandpass filter

$$H(e^{j\theta}) = \begin{cases} 1 & \text{if } 0.25\pi < |\theta| < 0.75\pi \\ 0 & \text{otherwise.} \end{cases}$$

Derive the impulse response h_k that corresponds to $H(e^{j\theta})$ by antitransform. Combine it with a rectangular window of 11 samples and derive the final transfer function $G(z)$. What are the advantages of using a wider window (more samples)? [25%]

(b) Discrete Fourier transform (DFT) and convolution.

The output of a FIR filter is efficiently computed via fast Fourier transform algorithms, Consider the FIR filter of 11 samples derived above and a 100-point DFT hardware.

(i) Write down the definitions of standard convolution and of circular convolution (related to a 100-point DFT and to a FIR filter of 11 samples). Define the range in which the two convolutions give the same output samples. [10%]

(ii) Explain how to use the DFT to compute the response of the filter to a sequence x_k of 200 samples. [30%]

(iii) Write the expression for the first sample of the DFT of the impulse response of the FIR filter. [10%]

4 (a) What does it mean for a random process to be *wide sense stationary* (WSS)? [10%]

(b) Define the autocorrelation function r_{XX} and the power spectral density (PSD) $\mathcal{S}_X$ of a WSS process $X(t)$. [10%]

(c) Let $X(t)$ be a smooth (infinitely differentiable) WSS process. Consider the process

$$Y(t) = \frac{d}{dt}X(t)$$

(i) Show that the cross correlation, $r_{XY}(t_1,t_2) = \frac{\partial}{\partial t_2} r_{XX}(t_1,t_2)$ [10%]

(ii) Show that $r_{YY}(t_1,t_2) = \frac{\partial}{\partial t_1} r_{XY}(t_1,t_2)$ [10%]

(iii) Hence derive an expression for r_{YY} in terms of r_{XX} and show that $Y(t)$ is WSS. [20%]

(iv) Suppose the PSD of X is $\mathcal{S}_X$. What is the PSD of Y? [10%]

(v) Define a new random process Y_τ as the output of a system H with input $X(t)$, where H is defined by

$$Y_\tau(t) = H[X(t)] = \frac{X_{t+\tau} - X_{t-\tau}}{2\tau}$$

for some positive constant τ. Write down the impulse response of H and use this to compute the transfer function, $\mathcal{H}(\omega)$. Compute the limit

$$\lim_{\tau \to 0} \mathcal{H}(\omega)$$

and explain the relevance of this limit to the process $Y(t)$ defined above. [Hint: you may use the Taylor expansion of $\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$] [30%]

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