EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 26 April 2022 9.30 to 11.10

Module 3F1

SIGNALS & SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Fig. 1 contains the Bode diagrams against θ (where $z = e^{j\theta}$) and the positive frequency part of the Nyquist diagrams of the systems

$$H_1(z) = 0.1\frac{z+1}{z-0.8}$$
$$H_2(z) = 0.3244(1 - 0.9\sqrt{2}z^{-1} + 0.81z^{-2})$$
$$H_3(z) = \frac{0.5372}{1 - 0.9\sqrt{2}z^{-1} + 0.81z^{-2}}$$

in shuffled order. Assign the correct Bode and Nyquist diagrams to each of those 3 systems and justify your choice. [30%]

(b) Sketch the complete Nyquist diagrams for the 3 systems. [15%]

(c) If the three systems are to be used in closed loop control systems, specify the stability range for the controller gain K for each of the three systems. An approximate answer based on a visual inspection of the Nyquist diagram is acceptable for this question. [20%]

(d) What is the steady state response of $H_1(z)$ to a unit step $u_k = 1$ for $k \geq 0$ and to the signal $v_k = (-1)^k$ for $k \geq 0$? [10%]

(e) The transfer function $H_1(z)$ was obtained from an analog system using the bilinear transformation $s \rightarrow a\frac{z-1}{z+1}$ where $a > 0$ was selected to ensure that the 3dB power attenuation frequency of the digital filter with samping period $T = 1\,\text{ms}$ is exactly the same as the 3dB frequency of the original continuous time system. Find a and the transfer function of the continuous time system in the Laplace domain. [25%]

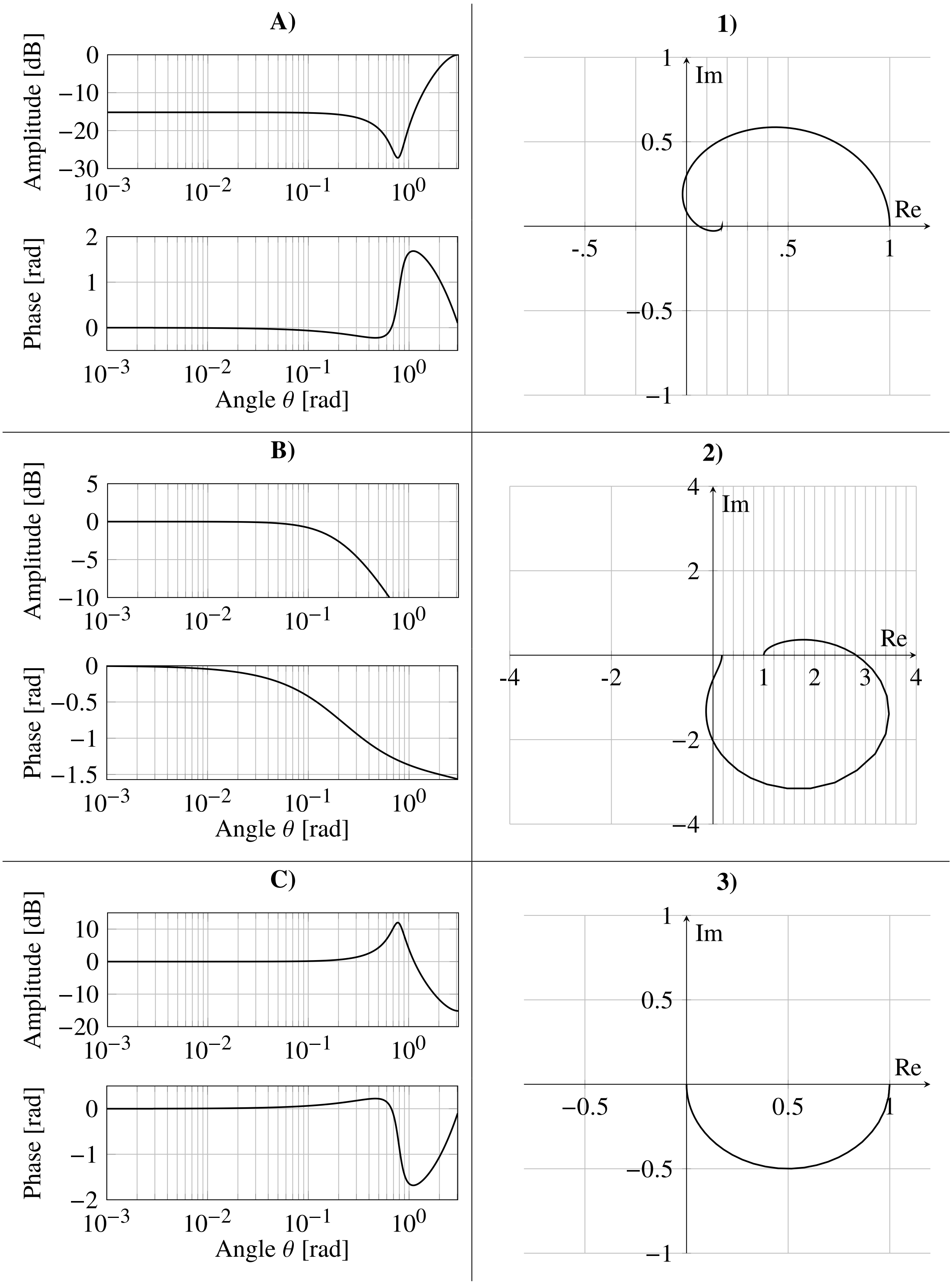
A)
Amplitude [dB]
0
−10
−20
−30
10^{-3} 10^{-2} 10^{-1} 10^{0}
Phase [rad]
2
1
0
10^{-3} 10^{-2} 10^{-1} 10^{0}
Angle θ [rad]
1)
1
Im
0.5
Re
-.5
.5
1
−0.5
−1
B)
Amplitude [dB]
5
0
−5
−10
10^{-3} 10^{-2} 10^{-1} 10^{0}
Phase [rad]
0
−0.5
−1
−1.5
10^{-3} 10^{-2} 10^{-1} 10^{0}
Angle θ [rad]
2)
4
Im
2
Re
-4
-2
1
2
3
4
−2
−4
C)
Amplitude [dB]
10
0
−10
−20
10^{-3} 10^{-2} 10^{-1} 10^{0}
Phase [rad]
0
−1
−2
10^{-3} 10^{-2} 10^{-1} 10^{0}
Angle θ [rad]
3)
1
Im
0.5
Re
−0.5
0.5
1
−0.5
−1

Fig. 1

2 Consider a (possibly complex) periodic sequence $\{h_k\}_{k\geq 0}$ with period N, and consider the Discrete Fourier Transform $(H_0, \ldots, H_{N-1})$ of its first period $(h_0, \ldots, h_{N-1})$.

(a) Suppose $H_\ell = \alpha \neq 0$ and $H_n = 0$ for all $n \neq \ell$ for some given ℓ with $0 \leq \ell \leq N-1$. Show that $h_k = (\alpha/N)e^{j\frac{2\pi\ell k}{N}}$ for all $k \geq 0$. Hence show that the z-transform of $\{h_k\}_{k\geq 0}$ is given by

$$H(z) = \frac{\alpha/N}{1 - e^{j\frac{2\pi\ell}{N}} z^{-1}}$$

[25%]

(b) Show that if $H_0, \ldots, H_{N-1}$ has d non-zero terms, then $H(z)$ has d distinct poles. [10%]

(c) For $N = 6$, $(H_0, \ldots, H_5) = (-1, 1, 0, 0, 0, 1)$. Find the z-transform $H(z)$ of $\{h_k\}_{k\geq 0}$ and sketch its pole-zero diagram. [20%]

(d) Now consider a system with transfer function $G(z) = (1 - z^{-1})H(z)$ where $H(z)$ is as computed in (c). Roughly sketch $|G(e^{j\theta})|$ for θ from 0 to π, specifying the values around $\theta = 0, \pi/3$ and π. [15%]

(e) The system with transfer function $G(z)$ is not stable. Determine a bounded input signal that results in an unbounded output signal. [10%]

(f) The system with transfer function $G(z)$ is used with feedback as shown in Fig. 2 where K is a constant. Suppose the closed-loop transfer-function $F(z)$ from $U(z)$ to $Y(z)$ has impulse response $f_0, f_1, f_2, \ldots$. Determine f_0 as a function of K. [20%]

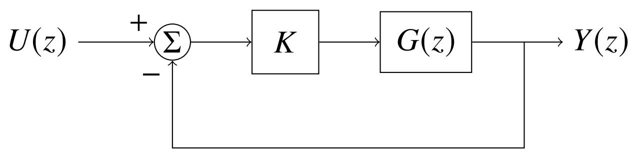

Fig. 2

3 (a) In a signal processing system it is desired to compute the Discrete Fourier Transform (DFT) of real valued data sequences, $\{x_n\}$, each of length N data points. The DFT is expressed in the standard format as:

$$X_k = \sum_{n=0}^{N-1} x_n \exp(-j2\pi nk/N)$$

Show that the DFT satisfies the conjugacy property, $X_k = X^*_{N-k}$, and a periodicity property $X_{k+mN} = X_k$ where m is any integer. Which, if any, of these properties applies when x_n is complex valued? [25%]

(b) Show that, for real x_n, the computational burden involved in direct calculation of the DFT is approximately N^2 real multiplications and N^2 real additions, for large N. You may assume that complex exponentials have been pre-computed and that memory accesses take negligible time. How does this compare with the burden when the data sequences are complex? [25%]

(c) Two real-valued sequences are labelled as $\{x_n^{(1)}\}$ and $\{x_n^{(2)}\}$ and it is desired to compute the individual DFTs $\{X_k^{(1)}\}$ and $\{X_k^{(2)}\}$. A scientist suggests that a speed-up can be obtained by combining this into a single complex sequence, $x_n = x_n^{(1)} + jx_n^{(2)}$ and taking the DFT of this complex sequence, $\{X_k\}$. By consideration of X_k and X^*_{N-k}, devise a suitable scheme for extraction of $\{X_k^{(1)}\}$ and $\{X_k^{(2)}\}$ from $\{X_k\}$. [30%]

(d) Determine the overall computational burden of the scheme in Part (c) and compare it with the direct calculation of $\{X_k^{(1)}\}$ and $\{X_k^{(2)}\}$. [20%]

4 (a) A linear continuous-time system has impulse response $h(t)$. The input to the linear system is a white noise process $\{x(t)\}$ which is a Wide Sense Stationary process with zero mean and autocorrelation function $r_{xx}(\tau) = \delta(\tau)$. Find the Power Spectral Density of the output of the linear system in the two cases:

$$\begin{aligned} \text{(i)}\ \ h(t) &= 2e^{-t}u(t), \\ \text{(ii)}\ \ h(t) &= -\delta(t) + 2e^{-t}u(t), \end{aligned}$$

where $u(t)$ is the Heaviside step function defined by: $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$. [35%]

(b) Let the pulse response of a causal digital filter be denoted by $\{h_k\}_{k\geq 0}$.

(i) Determine and sketch the frequency response of an FIR filter with:

$$h_0 = -1, \ \ h_1 = 2, \ \ h_2 = -1, \ \ h_k = 0 \text{ for } k \geq 3,$$

expressing your result in the form $H(e^{j\theta}) = \exp(-j\alpha\theta)G(\theta)$ where $G(\theta)$ is a real and positive function of θ and α is a constant to be determined. Explain whether the filter is high-pass or low-pass. [25%]

(ii) Consider the design of an 'all-pass' digital filter that passes all frequencies with unity gain but whose phase response may vary across frequency. Suppose that a first-order IIR filter is proposed, having a single pole at complex location p within the unit circle and single zero at location $1/p^*$. Show that such a filter would have constant gain across all frequencies. Propose how this filter might be adapted for use in a real-valued filtering scenario. Suggest an application for such an all-pass filter. [40%]

END OF PAPER