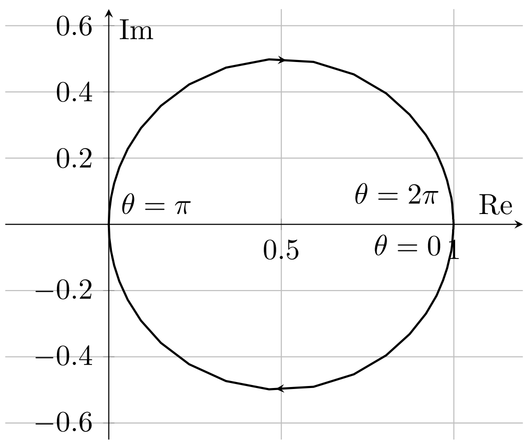
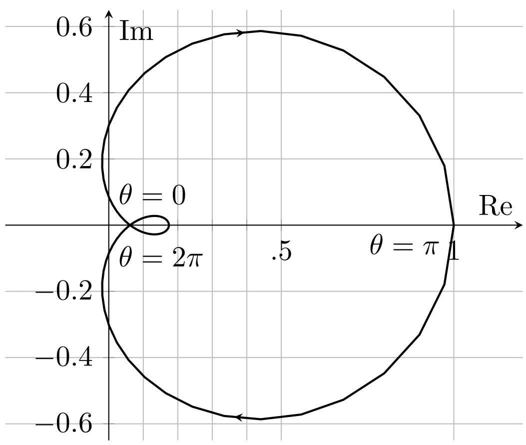
Crib of 3F1 exam 2022

1. (a)
   - $H_1(z)$ has a zero at -1 and a pole at 0.8. Since $H(1) = 1$ and $H(-1) = 0$, its amplitude plot should start at $0dB$ and end at $-\infty$ dB, which excludes Bode diagram (A). Its phase starts at 0 and should remain monotonically decreasing to $-pi/2$, excluding (C) where the phase clearly goes slightly positive. Hence, Bode diagram (B) corresponds to $H_1(z)$. Following the same logic, Nyquist diagram (3) is the only diagram whose phase never goes into the positive range.
   - $H_2(z)$ is an FIR filter with two zeros at $0.9e^{\pm j\pi/4}$ and two poles at the origin. Its amplitude reponse should experience a noted dip when it passes close to the zero at $\theta = \pi/4$, which is only the case for Bode diagram (A). Its phase diagram briefly exceeds $\pi/2$ around $\theta = 1$, which is only the case for Nyquist diagram (1).
   - By exclusion, $H_3(z)$ must correspond to Bode diagram (C) and Nyquist diagram (2). It is also clear that the phase and amplitude diagrams mirror that of the FIR filter $H_2(z)$ (save a bias due to a different mulltiplicative constant up front.) [30%]

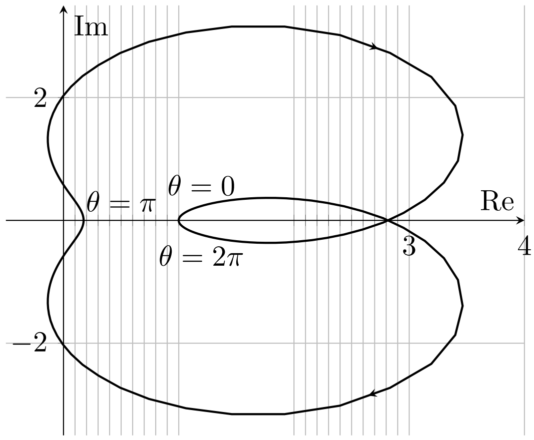
   (b) For $H_1(z)$,



For $H_2(z)$,



For $H_3(z)$,



[15%]

(c) All three systems are stable and hence have no unstable poles. The closed loop system is stable if there are no encirclements of the point $-1/K$. Hence

- For $H_1(z)$, the system is stable for $K > 0$ and for $K < -1$.
- For $H_2(z)$, the system is stable approxmately for $K < \frac{-1}{0.06} = -16.7$ and for $K > -1$. The exact answer is $H(e^{j\theta}) = 0.616$ for $\theta = \cos^{-1}(\sqrt{2}/1.8)$, i.e., $K < -16.2243$ but you were not expected to compute this.
- For $H_3(z)$, the system is stable approximately for $K < \frac{-1}{0.18} = -5.6$ and for $K > \frac{-1}{2.82} = -0.35$. Again, the exact answers are $H_3(e^{j\pi}) = H_3(-1) = 0.1743$, i.e., $K < -5.7386$, and $H_3(e^{j\theta}) = 2.8274$ for $\theta = \cos^{-1}(\sqrt{2}/1.8)$, i.e., $K > -0.3537$, but you were not expected to compute this. [20%]

(d) Reading from the Bode diagram, a unit step being a sinusoidal of zero frequency $u_k = \cos 0k$, the steady-state response will be a unit step of the same amplitude because $|H_1(e^{j0})| = |H_1(1)| = 1$. The steady step response to $v_k = (-1)^k = \cos \pi k$ is the all zero sequence because the gain of the system is zero at frequency π, i.e., $|H_1(e^{j\pi})| = |H_1(-1)| = 0$. [10%]

(e) We can invert the bilinear transformation to yield $z = \frac{a+s}{a-s}$ and insert this expression into $H_1(z)$ to obtain

$$\tilde{H}_1(s) = 0.1\frac{\frac{a+s}{a-s} + 1}{\frac{a+s}{a-s} - 0.8} = 0.1\frac{a + s + a - s}{a + s - 0.8a + 0.8s} = \frac{1}{1 + 9a^{-1}s}$$

This shows that the analog filter was of the form

$$H_1(s) = \frac{1}{1 + s/\omega_c},$$

a first-order lowpass where $\omega_c = a/9$ is the 3dB cutoff frequency since $10\log_{10}(|H_1(j\omega_c)|^2) = 10\log_{10}(1/2) \approx -3$. The discrete-time filter's 3dB cutoff frequency is obtained by solving

$$\left|0.1\frac{e^{j\theta} + 1}{e^{j\theta} - 0.8}\right|^2 = 1/2$$

which, after some manipulation, yields $\theta = \cos^{-1}(40/41) = 0.2213$ rad. (An approximate value of $\theta = 0.2$ obtained from the Bode plot is also satisfactory.) With a sampling period of $T = 1\,\mathrm{ms}$ the 3dB frequency should be $\omega_c = 221.3$ rad/sec which gives

$$H_1(s) = \frac{1}{1 + \frac{s}{221.3}}$$

and $a = 1992$. [25%]

2. (a) From the data book:

$$\begin{aligned} h_k &= \frac{1}{N}\sum_{n=0}^{N-1} H_n e^{j2\pi nk/N} \\ &= \frac{\alpha}{N} e^{j2\pi \ell k/N} \end{aligned}$$

for $0 \le k \le N-1$. But this expression is periodic with period N. Hence it is valid for all $k \ge 0$. Taking z-transforms gives:

$$\begin{aligned} H(z) &= \sum_{k=0}^{\infty} h_k z^{-k} \\ &= \frac{\alpha}{N}\sum_{k=0}^{\infty} e^{j\frac{2\pi \ell k}{N}} z^{-k} \\ &= \frac{\alpha/N}{1-e^{j2\pi\ell/N}z^{-1}}. \end{aligned}$$

[25%]

(b) This follows from the previous result by superposition due to the linearity of the DFT: a vector $(H_0, \ldots, H_{N-1})$ with d non-zero terms can be written as a sum of d vectors with one non-zero term, and hence the z transform $H(z)$ is a sum of d terms of the form

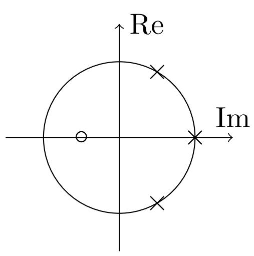
$$\frac{H_\ell/N}{1-e^{j2\pi\ell/N}z^{-1}}$$

for every non-zero term H_ℓ, which is a proper partial fraction expansion of a rational function with d distinct poles. [10%]

(c) We can either take the inverse DFT to obtain a period of the sequence in the time domain and take its z transform, or use the expressions and arguments in the previous two questions to write out the z transform directly as

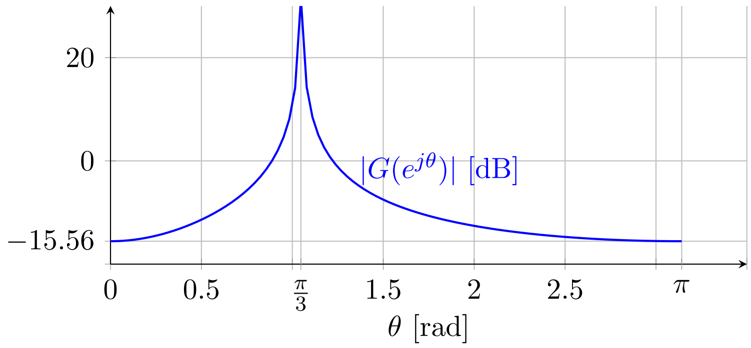
$$\begin{aligned} H(z) &= \frac{1}{6}\left(\frac{-1}{1-z^{-1}} + \frac{1}{1-e^{-j\pi/3}z^{-1}} + \frac{1}{1-e^{-j\pi/3}z^{-1}}\right) \\ &= \frac{-(1-2z^{-1}\cos\frac{\pi}{3}+z^{-2}) + (1-z^{-1})(2-2z^{-1}\cos\frac{\pi}{3})}{6(1-z^{-1})(1-e^{-j\pi/3}z^{-1})(1-e^{j\pi/3}z^{-1})} \\ &= \frac{1-2z^{-1}}{6(1-z^{-1})(1-e^{-j\pi/3}z^{-1})(1-e^{j\pi/3}z^{-1})} \end{aligned}$$

and the pole-zero diagram is hence



[20%]

(d) The amplitude plot has value 1/6, or, equivalently, -15.56 dB for $\theta = 0$ or π and tends to infinity as θ approaches $\pi/3$,



[15%]

(e) A sinusoidal with frequency $\theta = \pi/3$, e.g., $x_k = \sin(k\pi/3)$ for $k = 0, 1, 2, \ldots$, will result in an unbounded output sequence because $|G(e^{j\theta}|$ tends to infinity when θ approaches $\pi/3$ and the "steady state" term of the output sequence is unbounded (bearing in mind that the other terms do not necessarily decay). [10%]

(f) The transfer function of the open loop is $G(z) = \frac{1-2z^{-1}}{6(1-z^{-1}+z^{-2})}$ and hence the transfer function of the closed loop is

$$F(z) = \frac{KG(z)}{1+KG(z)} = \frac{K(1-2z^{-1})(1-z^{-1}+z^{-2})}{6(1-z^{-1}+z^{-2})+K(1-2z^{-1})} = \frac{K(1-3z^{-1}+3z^{-2}-2z^{-3})}{6+K-(6+2K)z^{-1}+6z^{-2}}$$

We can apply the initial value theorem

$$f_0 = \lim_{z\to\infty} F(z) = \frac{K}{6+K} = \frac{1}{1+6/K}.$$

[20%]

3. (a) From the definition:

$$\begin{aligned} X_{N-k} &= \sum_{n=0}^{N-1} x_n \exp(-j2\pi n(N-k)/N) \\ &= \sum_{n=0}^{N-1} x_n \exp(-j2\pi n(-k)/N). \end{aligned}$$

Hence

$$X^*_{N-k} = \sum_{n=0}^{N-1} x_n \exp(-j2\pi nk/N) = X_k$$

since x_n is real. This does not hold if x_n is *not* real. Also,

$$\begin{aligned} X_{k+mN} &= \sum_{n=0}^{N-1} x_n \exp(-j2\pi n(k+mN)/N) \\ &= \sum_{n=0}^{N-1} x_n \exp(-j2\pi nk/N) = X_k \end{aligned}$$

when m is an integer. This continues to hold if x_n is complex. [25%]

(b) Each term $x_n \exp(-j2\pi nk/N)$ requires 2 real multiplications.

Thus, for eack k, $2N$ real multiplications are needed to evaluate X_k as well as $2(N-1) \approx 2N$ real additions to sum the real and imaginary parts separately.

But, note that $X_k = X^*_{N-k}$ implies that we only need to calculate the first $N/2$ frequency values since the rest are obtained by simple conjugation.

Thus the total cost is N^2 real multiplications and N^2 real additions.

For complex data we require 4 real multiplies for each $x_n \exp(-j2\pi nk/N)$.

Thus, for each k, we require $4N$ real multiplications and $2N$ real additions.

But this time we need all N frequency components, so overall: $4N^2$ real multiplications and $2N^2$ real additions. [25%]

(c) By definition of x_n:

$$X_k = X_k^{(1)} + jX_k^{(2)}.$$

Hence

$$\begin{aligned} X_{N-k} &= X_{N-k}^{(1)} + jX_{N-k}^{(2)} \\ &= X_k^{(1)^*} + jX_k^{(2)^*} \end{aligned}$$

by the conjugacy property from (b). So,

$$X^*_{N-k} = X_k^{(1)} - jX_k^{(2)}$$

Hence,

$$X_k^{(1)} = 0.5(X_k + X_{N-k}^*)$$

and

$$X_k^{(2)} = -0.5j(X_k - X_{N-k}^*)$$

[25%]

(d) Complexity of X_k is $4N^2$ mults. and $2N^2$ additions.

Extraction of each k is 2 real additions for each of $X_k^{(1)}$ and $X_k^{(2)}$.

This is carried out $N/2$ times for the required $N/2$ coefficients of the real DFT.

So in total: $4N^2$ mults. and $2N^2 + N/2 \times 2 = 2N^2 + N$ additions.

Compared with evaluation of 2 real DFTs: $2N^2$ mults and $2N^2$ addns. Thus we do not benefit compared to an optimised real-valued DFT for each real sequence as in part (b). [But we would benefit if general complex DFT was being used throughout]. [20%]

4. (a) The power spectral density, $S_X(\omega)$, is given by

$$S_X(\omega) = \int_{-\infty}^{\infty} r_{XX}(\tau) e^{-j\omega\tau} d\tau = 1$$

which is constant over all ω. For the output of the linear system $y(t)$

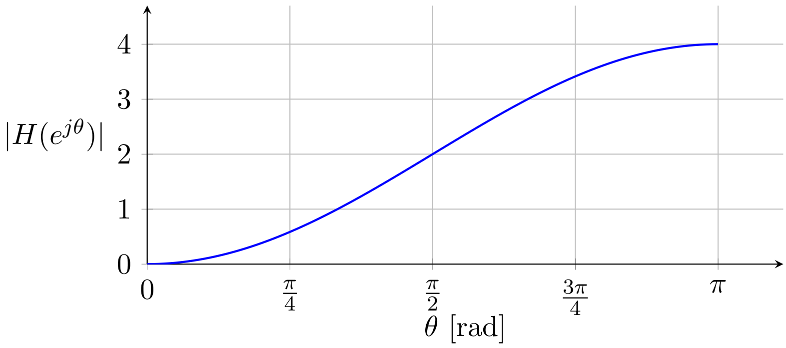
$$S_Y(\omega) = |H(j\omega)|^2 S_X(\omega)$$

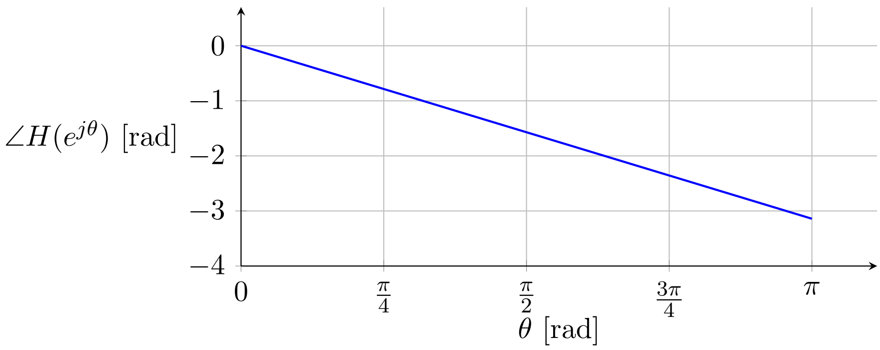
where $H(j\omega)$ is the Fourier transform of $h(t)$.
In case (i) $H(j\omega) = \dfrac{2}{1+j\omega}$. Hence $S_Y(\omega) = \dfrac{2}{(1+\omega^2)^{1/2}}$.
In case (ii) $H(j\omega) = -1 + \dfrac{2}{1+j\omega} = \dfrac{1-j\omega}{1+j\omega}$. Hence $S_Y(\omega) = 1$ for all ω. [In this case the transfer-function of the linear system is *all-pass*]. [35%]

(b) (i) $H(z) = -1 + 2z^{-1} - z^{-2}$ which gives:

$$\begin{aligned} H(e^{j\theta}) &= -1 + 2e^{-j\theta} - e^{-2j\theta} \\ &= e^{-j\theta}(-e^{j\theta} + 2 - e^{-j\theta}) \\ &= e^{-j\theta}(-2\cos\theta + 2) \end{aligned}$$

which gives $\alpha = 1$ and $G = 2 - 2\cos\theta$. This is a high-pass filter with a constant delay of 1 sample.





[25%]

(ii) The transfer function of the filter takes the form

$$H(z) = \alpha \frac{1 - z^{-1}/p^*}{1 - z^{-1}p}$$

for some constant α which gives

$$
\begin{aligned}
H(e^{j\theta}) &= \alpha \frac{1 - e^{-j\theta}/p^*}{1 - e^{-j\theta} p} \\
&= \frac{-\alpha e^{-j\theta}}{p^*} \frac{(1 - e^{-j\theta} p)^*}{1 - e^{-j\theta} p}.
\end{aligned}
$$

Hence

$$
|H(e^{j\theta})| = \left| \frac{\alpha}{p} \right|
$$

which is constant. Thus a digital all-pass function may take the form:

$$
H(z) = \frac{z^{-1} - p^*}{1 - z^{-1} p}.
$$

A problem for real implementation is that the filter doesn't have real coefficients. To overcome this problem it could be placed in series with a similar filter with poles and zeros at the conjugate locations:

$$
H(z) = \frac{z^{-1} - p^*}{1 - z^{-1} p} \frac{z^{-1} - p}{1 - z^{-1} p^*}
$$

which can be multiplied out to give a filter with real coefficients.
Application in cascades of 2nd order filters for phase effects in digital audio. [40%]