Crib of 3F1 exam 2023

May 23, 2023

1. (a) We solve the first difference equation by writing it in the z domain

$$zA(z) - a_0 z = \beta A(z)$$

and hence

$$A(z) = \frac{a_0 z}{a - \beta} = \frac{a_0}{1 - \beta z^{-1}}$$

giving (from the data book)

$$a_k = a_0 \beta^k \text{ for } k \geq 0.$$

(b) We transform the second difference equation into the z domain to obtain

$$zB(z) - b_0 z = \alpha(zA(z) - a_0 z)) + \gamma A(z) + \beta B(z)$$

and hence

$$\begin{aligned} B(z) &= \frac{(b_0 - \alpha a_0)z}{z - \beta} + \frac{\alpha z + \gamma}{z - \beta} A(z) \\ &= \frac{(\alpha a_0 - \alpha a_0)z}{z - \beta} + \frac{(\alpha z + \gamma) a_0 z}{(z - \beta)^2} \\ &= \frac{(\alpha z + \gamma) a_0 z}{(z - \beta)^2} \end{aligned}$$

(c) Using the initial value theorem, $b_0 = \lim_{z \to \infty} B(z) = \alpha a_0$, which is consistent with the second equation.

(d) The z transform of a geometric sequence q^k is

$$\frac{1}{1 - qz^{-1}} = 1 + qz^{-1} + q^2 z^{-2} + \ldots.$$

Taking the derivative with respect to q on both sides of this equation, we obtain

$$\frac{z^{-1}}{(1 - qz^{-1})^2} = \frac{z}{(z - q)^2} = z^{-1} + 2qz^{-2} + 3q^2 z^{-3} + 4q^3 z^{-4} + \ldots$$

which, from examining the coefficients of the power series in z^{-1}, is the z transform of the sequence $\{kq^{k-1}\}_{k \geq 0}$.

(e) We note that

$$B(z) = \alpha a_0 \frac{z^2}{(z-\beta)^2} + +\gamma a_0 \frac{z}{(z-\beta)^2}$$

hence

$$b_k = \alpha a_0 (k+1)\beta^k + \gamma a_0 k \beta^{k-1} = a_0 \left[\alpha\beta(k+1) + \gamma k\right] \beta^{k-1}$$

and

$$c_k = (1-\beta) b_k = (1-\beta) a_0 \left[\alpha\beta(k+1) + \gamma k\right] \beta^{k-1}$$

(f) This is easiest done in the z domain because

$$\begin{aligned}
\sum_{k=0}^{\infty} (a_k + b_k + c_k) &= A(1) + B(1) + C(1) \\
&= \frac{a_0}{1-\beta} + (1 + (1-\beta)) \frac{(\alpha+\gamma) a_0}{(1-\beta)^2} \\
&= \frac{a_0}{1-\beta} + (2-\beta) \frac{(1-\beta) a_0}{(1-\beta)^2} \\
&= \frac{a_0 (3-\beta)}{1-\beta} = 1
\end{aligned}$$

and hence

$$a_0 = \frac{1-\beta}{3-\beta}.$$

(g) For $\beta = 1/2$ and $\alpha = \gamma = 1/4$, $a_0 = 1/5$,

$$c_k = \frac{1}{10} \left[\frac{1}{8}(k+1) + \frac{1}{4} k\right] \left(\frac{1}{2}\right)^{k-1} = \frac{1}{40} (3k+1) 2^{-k}$$

and hence

$$\begin{aligned}
E[C|\text{game over}] &= \frac{\sum_{k=0}^{\infty} k c_k}{\sum_{k=0}^{\infty} c_k} \\
&= \frac{3 \sum_{k=0}^{\infty} k^2 2^{-k} + \sum_{k=0}^{\infty} k 2^{-k}}{3 \sum_{k=0}^{\infty} k 2^{-k} + \sum_{k=0}^{\infty} 2^{-k}}
\end{aligned}$$

To evaluate those two sums we use the same derivation trick we used in part (d) on the sum of a geometric sequence, namely,

$$\begin{cases}
\frac{1}{1-q} & = \sum_{k=0}^{\infty} q^k \\
\frac{1}{(1-q)^2} & = \sum_{k=0}^{\infty} k q^{k-1} \\
\frac{q}{(1-q)^2} & = \sum_{k=0}^{\infty} k q^k \\
\frac{1+q}{(1-q)^3} & = \sum_{k=0}^{\infty} k^2 q^{k-1} \\
\frac{q(1+q)}{(1-q)^3} & = \sum_{k=0}^{\infty} k^2 q^k
\end{cases}$$

and hence

$$\begin{cases} \sum_{k=0}^{\infty} 2^{-k} = 2 \\ \sum_{k=0}^{\infty} k 2^{-k} = 2 \\ \sum_{k=0}^{\infty} k^2 2^{-k} = 6 \end{cases}$$

and so

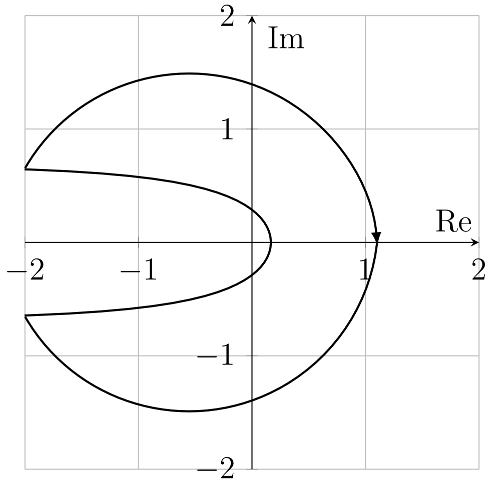
$$E[C|\text{game over}] = \frac{3 \times 6 + 2}{3 \times 2 + 2} = \frac{5}{2} = 2.5$$

so the expected return (3 cookies) is more than the expected loss (2.5 cookies) but personally I wouldn't take a risk for an expected return of half a cookie and would hence just eat the cookies I have.

2. (a)

$$G(z) = \frac{1}{6\left(\frac{z-1}{z+1}\right)^2} \cdot \frac{\left(1+\frac{z-1}{z+1}\right)^3}{\frac{z-1}{z+1}+\frac{1}{3}}$$
$$= \frac{(z+1+(z-1))^3}{2(z-1)^2(3(z-1)+z+1)}$$
$$= \frac{4z^3}{(z-1)^2(4z-2)}$$
$$= \frac{1}{(1-z^{-1})^2\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2}{(1-2z^{-1}+z^{-2})(2-z^{-1})}$$

(b) The complete Nyquist diagram is



Since there are no open-loop poles outside the unit circle, the Nyquist stability criterion requires no encirclements of $-1/K$ to achieve stability. This applies for $-1/K < G(e^{j\pi}) = 1/6$ hence $K > 0$ or $K < -6$ results in a stable system.

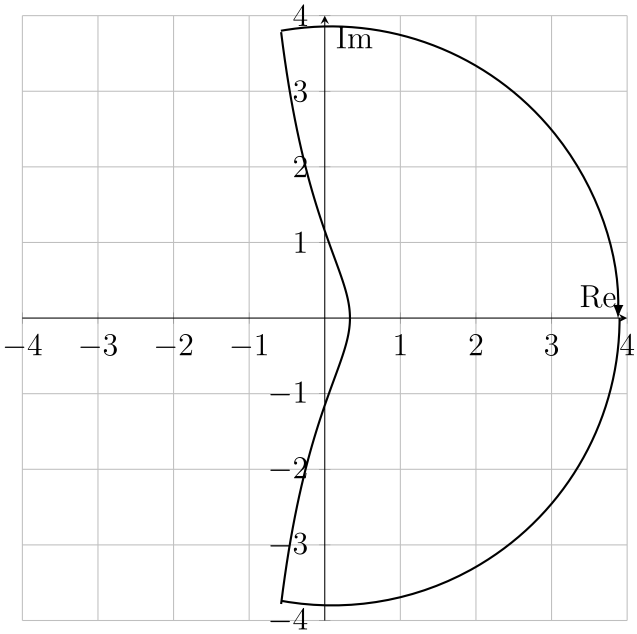
(c) The z transform of the difference equation is $Y(z) = (1 - z^{-1})X(z)$ hence the transfer function of the FIR is $(1 - z^{-1})$, resulting in an overall transfer function

$$H(z) = \frac{1}{(1-z^{-1})\left(1-\frac{1}{2}z^{-1}\right)} = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)}$$

(d) The final value theorem applies here because the system has a single pole at 1 and the remaining pole inside the unit circle. Hence,

$$\lim_{k\to\infty} h_k = \lim_{z\to 1}(z-1)H(z)$$
$$= \lim_{z\to 1}\frac{z^3}{z-\frac{1}{2}}$$
$$= 2$$

(e) (i) The complete Nyquist diagram is



Since there are no open-loop poles outside the unit circle, the Nyquist stability criterion requires no encirclements of $-1/K$ to achieve stability. This applies for $-1/K < H(e^{j\pi}) = 1/3$ hence $K > 0$ or $K < -3$ results in a stable system.

(ii) We can take polynomial approximations of exponentials as

$$\begin{aligned}
H(e^{j\theta}) &= \frac{1}{\left(1-e^{-j\theta}\right)\left(1-\frac{1}{2}e^{-j\theta}\right)} \\
&= \frac{1}{\left(1-\left[1+(-j\theta)+\frac{(-j\theta)^2}{2}+o(\theta^3)\right]\right)\left(1-\frac{1}{2}\left[1+(-j\theta)+o(\theta^2)\right]\right)} \\
&= \frac{1}{\left(j\theta-\frac{(j\theta)^2}{2}+o(\theta^3)\right)\left(\frac{1}{2}+\frac{j\theta}{2}+o(\theta^2)\right)} \\
&= \frac{2}{j\theta}\cdot\frac{1}{\left(1-j\frac{\theta}{2}+o(\theta^2)\right)\left(1+j\theta+o(\theta^2)\right)} \\
&= \frac{2}{j\theta}\left(1+j\frac{\theta}{2}+o(\theta^2)\right)\left(1-j\theta+o(\theta^2)\right) \\
&= \frac{2}{j\theta}\left(1-\frac{j\theta}{2}+o(\theta^2)\right) \\
&= -1-j\frac{2}{\theta}+o(\theta)
\end{aligned}$$

3. (a) (i) $\mathbf{x_1}$ and $\mathbf{x_2}$ both satisfy the symmetry condition $x_k = x^\star_{N-k}$ and hence their DFT is one of the two real vectors $\mathbf{X_c}$ and $\mathbf{X_d}$. X_c satisfies the symmetry condition $X_k = X^\star_{N-k}$ and hence its inverse DFT is real, while $\mathbf{X_d}$ does not and hence its inverse DFT must be complex. This determines that $\mathbf{X_c}$ corresponds to $\mathbf{x_1}$ while $\mathbf{X_d}$ corresponds to $\mathbf{x_2}$.

The remaining two vectors $\mathbf{x_3}$ and $\mathbf{x_4}$ do not satisfy the symmetry condition and hence their DFT must be one of the two complex solutions $\mathbf{X_a}$ and $\mathbf{X_b}$. We note that the first coefficient of the DFT is the sum of the components of the time domain vector, which is $3(u+w)$ for $\mathbf{x_1}, \mathbf{x_2}$ and $\mathbf{x_3}$, and $4u+2w$ for $\mathbf{x_4}$. Since $\mathbf{X_b}, \mathbf{X_c}$ and $\mathbf{X_d}$ have the same first coefficient 6, the odd one out $\mathbf{X_a}$ must correspond to $\mathbf{x_4}$ while $\mathbf{X_b}$ corresponds to $\mathbf{x_3}$ by exclusion of all other pairings.

To summarise, the pairings are $(\mathbf{x_1}, \mathbf{X_c}), (\mathbf{x_2}, \mathbf{X_d}), (\mathbf{x_3}, \mathbf{X_b})$ and $(\mathbf{x_4}, \mathbf{X_a})$.

(ii) Following the argument above, we get the set of equations

$$\begin{cases} 3u + 3w = 6 \\ 4u + 2w = 2 \end{cases}$$

which has a unique solution $(u, w) = (-1, 3)$.

(b) (i) The audio signal is unbiased, i.e., its mean is 0. While we cannot be sure that the mean will be zero on each block of 1024 symbols, it will be close to zero. Since the first element X_0 of the DFT is the sum of the time domain symbols, we expect it to be close to zero.

(ii) Since the signal is real valued, its DFT satisfies the symmetry condition, hence $X_{1024-37} = X_{987} = X^\star_{37} = 4 - 3j$.

(iii) The real-world frequency is

$$\frac{k}{N} f_s = \frac{37}{1024} 20 \times 10^3 = 722.66 \text{ Hz}$$

(iv) The FFT has complexity $O(N \log N)$ whereas direct computation of the DFT by matrix multiplication has complexity $O(N^2)$. The outcome does not differ in any way from the outcome of a DFT as the FFT is just a re-writing of the DFT operations in a manner that takes advantage of the specific DFT matrix structure. In the specific example, computation via the FFT results in approximately $10 \times 2^{10} = 10,240$ operations whereas direct computation of the DFT results in $2^{20} \approx 10^6$, i.e., a million operations.

(v) The same outcome can be obtained from cutting out the middle half of the DFT. Specifically, the new DFT vector should be

$$\mathbf{X}' = [X_0, X_1, \ldots, X_{255}, x, X_{769}, \ldots, X_{1023}]$$

where the middle value x can be set to 0: it represents the frequency component at $f'_s/2$ which is zero if the original signal had a bandwith of 4 kHz.

(vi) The effect of the shift will be a multiplication in the frequency domain by $e^{jn\theta}$ where $\theta = 2\pi/1024$ for the 10th block, corresponding to a drift of one sample. In this discussion, we neglect the marginal effect of the difference between a cyclic shift and a linear shift: the phase effect described corresponds to a cyclic shift where the first sample of the block has been shifted to the back of the block, whereas the actual sampling drift will result in a linear shift where the last sample of the 10th block is what should have been the first sample of the 11th block. The difference between those two cases is negligible as only one sample is involved. The shift can be compensated by applying a multiplication by $e^{jn\pi/5120}$ to every DFT vector computed, corresponding to a cyclic shift of one tenth of a sample.

4. (a) The variance is $r_{XX}(0)$ or the inverse Fourier transform of the power spectral density evaluated at $\tau = 0$, i.e.,

$$\begin{aligned}
\sigma_X^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega 0}d\omega \\
&= \frac{KL}{\pi}\int_{-\infty}^{\infty}\frac{1+12L^2\omega^2}{(1+4L^2\omega^2)^2}d\omega \\
&= \frac{KL}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1+3\tan^2\theta}{(1+\tan^2\theta)^2}(1+\tan^2\theta)\frac{d\theta}{2L} \\
&= \frac{K}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{1+3\tan^2\theta}{1+\tan^2\theta}d\theta \\
&= \frac{K}{2\pi}\int_{-\pi/2}^{\pi/2}(1+2\sin^2\theta)d\theta \\
&= \frac{K}{2\pi}\int_{-\pi/2}^{\pi/2}(2-\cos 2\theta)d\theta \\
&= \frac{K}{2\pi}\left[2\theta - \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = K
\end{aligned}$$

where in step 2 we substituted $\tan\theta = 2L\omega$ and hence $(1+\tan^2\theta)d\theta = 2Ld\omega$.

(b)

$$\begin{aligned}
S_X(\omega) &= 2KL\frac{1-(j\sqrt{12}L\omega)^2}{(1-(j4L\omega)^2)^2} \\
&= \sqrt{2KL}\frac{1+j\sqrt{12}L\omega}{(1+j\sqrt{4}L\omega)^2}\sqrt{2KL}\frac{1-j\sqrt{12}L\omega}{(1-j\sqrt{4}L\omega)^2} \\
&= \left|\sqrt{2KL}\frac{1+j2\sqrt{3}L\omega}{(1+j2L\omega)^2}\right|^2
\end{aligned}$$

where the last step follows from the fact that a complex number times its conjugate equals its magnitude squared.

(c) As shown in the lecture, a variance $\mathrm{Var}(W_k) = 1/h$ achieves $\lim_{h\to 0}\frac{1}{T}E\left[|U_h(j\omega)|^2\right] = 1$ for all ω and hence $\lim_{h\to 0}\frac{1}{T}E\left[|Y_h(j\omega)|^2\right] = |H(j\omega)|^2$.

(d) (i) The step invariant transform is

$$H(z) = (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\frac{H(s)}{s}\right\}_{t=kh}$$

where $H(s) = \sqrt{2KL}\frac{1+2\sqrt{3}Ls}{(1+2Ls)^2}$ as shown in (b) where we have replaced $j\omega$ by the Laplace variable s and used the sampling intervals h as given.

We use partial fractions to invert the Laplace transform

$$\begin{aligned}\frac{H(s)}{s} &= \sqrt{2KL}\left[\frac{A}{s} + \frac{B}{1+2Ls} + \frac{C}{(1+2Ls)^2}\right] \\ &= \sqrt{2KL}\cdot\frac{A + 4ALs + 4AL^2s^2 + Bs + 2BLs^2 + Cs}{s(1+2Ls)^2}\end{aligned}$$

yielding

$$\begin{cases} A = 1 \\ 4AL + B + C = 2\sqrt{3}L \\ 4AL^2 + 2BL = 0 \end{cases}$$

and hence $A = 1$, $B = -2L$ and $C = 2(\sqrt{3}-1)L$, giving

$$\frac{H(s)}{s} = \frac{\sqrt{2}}{s} + \frac{5\sqrt{2}}{s+5} + \frac{5(\sqrt{3}-1)}{(s+5)^2}$$

where we have inserted the given values $K = 10$ and $L = 0.1$, and $\mathcal{H}(t)$ is the Heaviside step function $\mathcal{H}(t) = 0$ for $t < 0$ and $\mathcal{H}(t) = 1$ for $t \geq 0$. We thus obtain the continous time impulse response

$$h(t) = \sqrt{2}\mathcal{H}(t) + 5\sqrt{2}e^{-5t} + 5(\sqrt{3}-1)te^{-5t}$$

and sample it at intervals h to obtain the sequence

$$g_k = \sqrt{2} + 5\sqrt{2}e^{-5kh} + 5(\sqrt{3}-1)khe^{-5kh} \text{ for } k \geq 0.$$

We now move to the z domain to obtain the discrete-time transfer function by multiplying the z transform of the sequence g_k (which is the step-response of the system) by $(1-z^{-1})$ to obtain the transfer function

$$\begin{aligned}H(z) &= (1-z^{-1})G(z) \\ &= (1-z^{-1})\left[\frac{\sqrt{2}}{1-z^{-1}} + \frac{5\sqrt{2}}{1-e^{-5h}z^{-1}} + \frac{5(\sqrt{3}-1)e^{-5h}z^{-1}}{(1-e^{-5h}z^{-1})^2}\right] \\ &= \sqrt{2} + 5\sqrt{2}\frac{z-1}{z-e^{-5h}} + 5(\sqrt{3}-1)e^{-5h}\frac{z-1}{(z-e^{-5h})^2}\end{aligned}$$

(ii) The power spectral density has a double pole at the frequency $\omega = \frac{1}{2L}$ rad $\cdot$ s^{-1}. The digital model covers frequencies up to half the sampling frequency, i.e., $f_s/2 = \frac{1}{2h}$ Hz for a sampling interval h. For the system to be accurate, we must have

$$\frac{1}{2\pi}\cdot\frac{1}{2L} \ll \frac{1}{2h}$$

and hence $h \ll 2\pi L = 0.62$ s for $L = 0.1$. Hence, a sampling interval of, say, $h = 10$ms would be a good choice.
The step invariant response is appropriate in this case because in the derivation of the convergence of the power spectral density of the system output alluded to in part (c), i.e., $\lim_{h\to 0} \frac{1}{T} E\left[|Y_h(j\omega)|^2\right] = |H(j\omega)|^2$, a piecewise constant signal with constant intervals of length h was assumed, and hence the resulting continuous time filter output is a superposition of step responses. Modeling the digital system using the step-invariant transformation ensures that the digital signal obtained at the output of the digital filter is equivalent to sampling the output of the continuous time filter, so its statistical properties and in particular the scaling with the variance of the variables W_k are maintained.

Engineering Tripos Part IIA, 2023
Assessor's Report
Module 3F1: Signals and Systems

The examination was taken by 164 students, including 9 who did not have a mark for Part IB. The percentual mean mark was 62.38% and the standard deviation on this was 12.29.

Question 1: z Transform
Attempts: 161, mean 12.6/20, highest 19/20, lowest 4/20
This question covered applications of the z transform based on probabilistic analysis of a game (inspired by DNA alignment). There was some concern expressed by the external examiner and 2nd assessor that students might struggle with the probabilistic aspects of the question since 3F1 does not cover probabilities (but then neither does it cover many of the other applications of the z transform used as a basis for past questions). The question required no understanding of underlying probability because all relevant difference equations were given. The only probability knowledge students needed was how to compute an expectation for a given probability distribution. As it turns out, almost none of the students struggled with the probabilistic aspects of the question and all students took the question at face value, attempting to solve difference equations. What is more concerning is that a fairly large proportion of students failed to understand the link between the z transform and difference equations. Questions about the z transform were answered well, showing that all had understood the properties of the transform, but questions about difference equations were answered by some students by writing down instances of equations in the time domain for k=0,1,2,3… and trying to guess the recursion rule as one would have done at A levels, without using the z transform. This indicates a weakness in the examples papers that insufficiently emphasises the raison d'être for the z transform and will be addressed in next year's course by introducing new questions that highlight the use of the z transform for solving difference equations.

Question 2: Stability
Attempts: 165, mean 12.6/20, highest 20/20, lowest 4/20
This popular question was very much in line with past examples papers and generally answered well. All students knew how to complete a Nyquist diagram by mirroring the curve given for positive frequencies about the x axis, but many struggled when completing the curve with a loop "at infinity" when going from frequency 2π to 0. Some otherwise good students also forgot that poles on the unit circle can be considered inside the unit circle for the purpose of the Nyquist stability criterion (stability of the close loop inferred from the position of the poles in the open loop system) even though they would cause instability for the open loop. Overall, this was the question the students struggled with the least given that it was quite similar to past exam questions.

Question 3: Discrete Fourier Transform
Attempts: 17, mean 12.6/20, highest 18/20, lowest 3/20
This was a surprisingly unpopular question, given that it required far less algebra and hence less time than other questions. Those who did solve it came in three groups: about 10 students did very well on the question, 3 students were close to the average, and 4 students had very low scores that weighs the average down. The main reason for the question's

unpopularity is probably that the topic was presented in revised form in lectures this year, but the corresponding examples paper was not yet changed, giving students insufficient opportunity to practice what they had learned in lectures. This will be addressed in next year's course. There was a typo in this question that was reported after the exam had ended. It clearly caused confusion for 2 students, may have caused confusion for another 2, and clearly caused no confusion for the remaining 13 students. An annex to this report details the typo, its effect, remedial actions, and evidence collected in individual cases to decide whether the typo had caused any confusion.

Question 4: Random Processes
Attempts: 152, mean 12.1/20, highest 18/20, lowest 5/20
This question covered another part of the course that has been revised this year. A new question in the examples paper allowed students to gain more familiarity with the new way of introducing random processes and hence more students were willing to attempt this question. The question required quite a lot of algebra and was marked leniently as very few students succeeded in completing those calculations without errors, which is understandable under exam pressure. Many students only attempted a few parts of this question, so that despite lenient marking it has a lower average. This is probably an indication of bad time management rather than difficulty with the actual question, as many students would have done this question last if attempting questions in order of their numbering.

3F1 Examination 2023 – Report on handling of typo in Question 3

Introduction: question 3(a)(i) gave a list of 4 time-domain vectors ($\mathbf{x_1},\mathbf{x_2},\mathbf{x_3},\mathbf{x_4}$) and a list of 4 frequency-domain vectors ($\mathbf{X_a},\mathbf{X_b},\mathbf{X_c},\mathbf{X_d}$) and students were asked to determine which frequency-domain vector corresponds to which time-domain vector. This question could be solved easily based on two properties of the Discrete Fourier Transform (DFT):

1. Time-domain vectors satisfying the "conjugate symmetry property" have corresponding real-valued (as opposed to complex-valued) frequency-domain vectors, and vice versa, real-valued time-domain vectors correspond to frequency domain vectors satisfying the "conjugate symmetry property".
2. The first component "X_0" of a frequency-domain vector is simply the sum of all the components of the time-domain vector (note that it is common practice to index vectors from 0 to N-1 when working with the DFT).

Typo: it was intended that the frequency-domain vector $\mathbf{X_a}$ should satisfy the conjugate symmetry property and correspond to time-domain vector $\mathbf{x_4}$. However, a typo resulted in a change of sign in the imaginary part of one component and as a result this vector no longer obeyed the conjugate symmetry property.

Why this typo was not discovered by the assessors: all frequency-domain vectors except $\mathbf{X_a}$ had a first component of 6, whereas $\mathbf{X_a}$'s first component was 2. It was easy to immediately assign $\mathbf{X_a}$ to $\mathbf{x_4}$ by noticing that $\mathbf{x_4}$ is the time-domain vector whose sum was different from the other 3. Anyone who solves the problem this way never needs to further scrutinise $\mathbf{X_a}$ and would miss the fact that it didn't satisfy the conjugate symmetry property. This is also what the crib did.

How this typo was reported: nobody raised a question during the exam. A few days after the exam, a student went to view the exam paper at the department Teaching Office and reported the typo to the Director for Undergraduate Education.

Remedial actions:

- Every submitted script was carefully analysed for evidence that the typo caused confusion or delay.
- For those for whom there was evidence or even just a suspicion that it may have caused confusion or delay, tailored remedial action was taken as described for each individual case below.
- For all students, Question 3(b)(ii) further probed understanding of the conjugate symmetry property and was hence marked leniently. The correct answer was X_{987}=4-3j as per the conjugate symmetry property, but any candidate who forgot to conjugate (X_{987}=4+3j) or got the index slightly wrong (X_{986} or X_{988}=4-3j) was awarded full points for this question (1 point).
- A further version of the exam paper JS/5 will be produced and uploaded to "past exams" on the department website so as not to cause confusion to future revising students.

Statistics: Question 3 was the least popular question and was selected by only 18 out of 169 candidates (including one candidate who solved 4 questions and for whom Q3 was the weakest and hence not counted). There were only 2 scripts with clear evidence that the typo caused confusion or delay, and 2 scripts for which there is a sufficient basis to suspect that the typo may have caused confusion. For the remaining 14 scripts, it is either clear that the candidate followed the same solution path as the crib and never scrutinised $\mathbf{X}_a$ further, or there is sufficient evidence that the candidate has misunderstood DFT properties so as to exclude that they would have the ability to identify the typo.

[The following sections detail individual cases and remedial action taken for the 4 scripts affected and have been deleted from the anonymised report.]