Version JS/5[1]

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 25 April 2023 9.30 to 11.10

Module 3F1

SIGNALS & SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

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[1]Version JS/4 used in the exam had a typo in Question 3: the third component of $\mathbf{X_a}$ was erroneously stated as $-4 + 6.93j$. This version is the corrected exam paper.

1 A game has two stages: when in stage A, with probability α, a player jumps to stage B for free, with probability γ they jump to stage B at a cost of 1 cookie, or with probability β they waste 1 cookie and remain in stage A, where $\alpha + \beta + \gamma = 1$. Once in stage B, with probability β they waste 1 cookie and remain in stage B, while with probability $1 - \beta$ they finish the game and win 3 cookies. Figure 1 illustrates the game as a graph.

Let a_k be the probability of being in stage A and having spent k cookies, b_k the probability of being in stage B and having spent k cookies, and c_k the probability of exiting the game and having spent k cookies. These quantities obey the difference equations

$$\begin{cases} a_{k+1} & = \beta a_k \\ b_0 & = \alpha a_0 \\ b_{k+1} & = \alpha a_{k+1} + \gamma a_k + \beta b_k \\ c_k & = (1 - \beta) b_k \end{cases}$$

for $k \geq 0$.

(a) Express a_k for $k > 0$ in terms of a_0 and β. [10%]

(b) Show that the z transform of $\{b_k\}_{k \geq 0}$ satisfies $B(z) = \frac{(\alpha z + \gamma) a_0 z}{(z - \beta)^2}$. [20%]

(c) Determine b_0 from $B(z)$ and show that it is consistent with the second difference equation. [10%]

(d) By taking the derivative with respect to q of the z transform of a geometric sequence $\{q^k\}_{k \geq 0}$, show the the z transform of the sequence $\{k q^{k-1}\}_{k \geq 0}$ is $\frac{z}{(z - q)^2}$. [10%]

(e) Express c_k for $k \geq 0$ in terms of a_0, α, β and γ. [10%]

(f) Determine a_0 as a function of β, bearing in mind that the relevant probabilities sum to 1, i.e., $\sum_{k=0}^{\infty} (a_k + b_k + c_k) = 1$ and $\alpha + \beta + \gamma = 1$. [20%]

(g) For $\beta = 1/2$ and $\alpha = \gamma = 1/4$, compute the expected number of cookies spent once the game ends, $E[C \mid \text{game over}]$, and comment on whether the expected winnings exceed the expected loss for this set of parameters. [20%]

Note that the probability of having spent k cookies once the game ends is

$$P(k \text{ cookies} \mid \text{game over}) = \frac{P(k \text{ cookies AND game over})}{P(\text{game over})} = \frac{c_k}{\sum_{k=0}^{\infty} c_k}$$

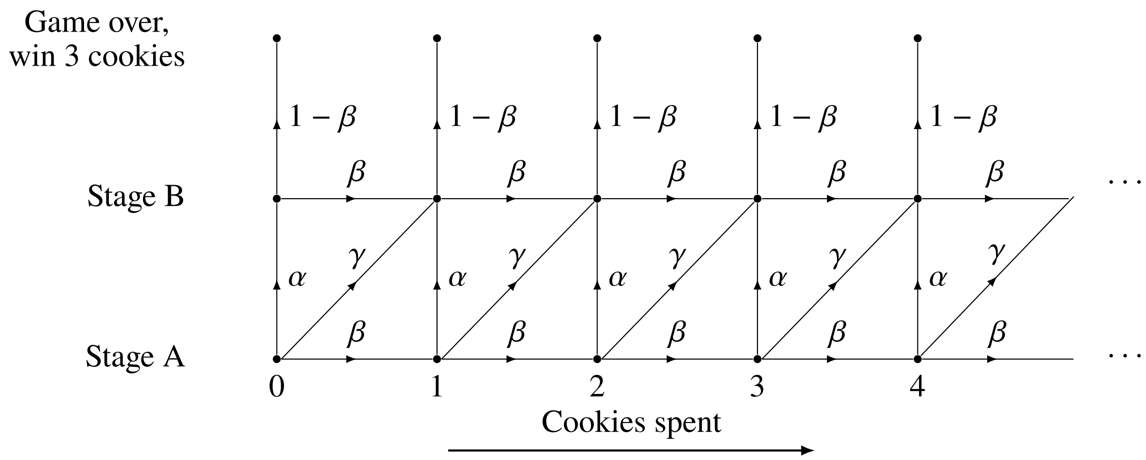
Game over,
win 3 cookies
Stage B
Stage A
$1-\beta$
β
α
γ
. . .
0
1
2
3
4
Cookies spent

Fig. 1

2 Consider the analogue system with the transfer function in the Laplace domain

$$G(s) = \frac{1}{6s^2} \cdot \frac{(1+s)^3}{s+\frac{1}{3}}$$

(a) Show that a digital model

$$G(z) = \frac{2}{(1-2z^{-1}+z^{-2})(2-z^{-1})}$$

of this analogue system can be obtained using the bilinear transform $s \to \frac{z-1}{z+1}$. [15%]

(b) Consider placing this system in a digital feedback control loop with gain K. A partial Nyquist diagram of the digital transfer function is given in Fig. 2(a) where the normalised frequency θ goes from 0 to π. Sketch the complete Nyquist diagram and determine for which values of K, if any, the feedback loop achieves stability. [20%]

(c) The sampled output $\{x_k\}_{k\geq 0}$ of the analogue system is fed into an FIR filter whose output sequence $\{y_k\}_{k\geq 0}$ satisfies the difference equation

$$y_k = x_k - x_{k-1}$$

where you can assume that $x_k = y_k = 0$ for $k < 0$. What is the combined system's transfer function $H(z)$? [10%]

(d) Consider the impulse response $\{h_k\}_{k\geq 0}$ of the combined system. Determine $\lim_{k\to\infty} h_k$. [15%]

(e) Consider placing the combined system in a digital feedback control loop with gain K. Part of the Nyquist diagram of the digital transfer function is given in Fig. 2(b).

(i) Sketch the complete Nyquist diagram and determine for which values of K, if any, the feedback loop achieves stability. [20%]

(ii) Derive the asymptote of the Nyquist diagram as the normalised frequency θ goes to zero. [20%]

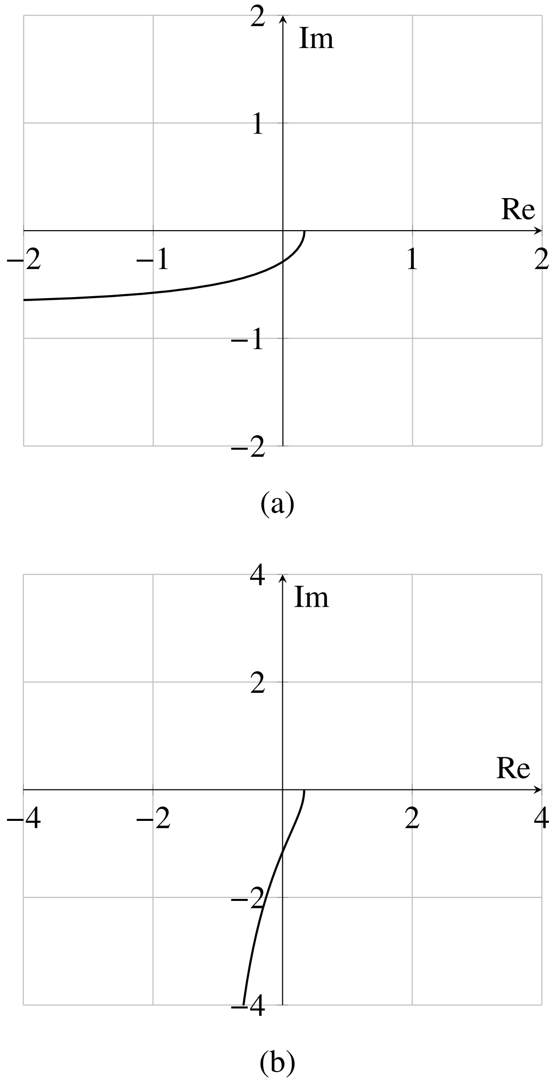

Fig. 2

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3 (a) Consider the following two lists of time domain vectors and of Discrete Fourier Transform (DFT) frequency domain vectors. u and w are non-zero real numbers.

Time domain	Frequency Domain
$\mathbf{x_1} = [u, u, w, w, w, u]$	$\mathbf{X_a} = [2, 0, -4 - 6.93j, 0, -4 + 6.93j, 0]$
$\mathbf{x_2} = [u, u + jw, w - ju, w, w + ju, u - jw]$	$\mathbf{X_b} = [6, -4 + 6.93j, 0, -4, 0, -4 - 6.93j]$
$\mathbf{x_3} = [u, u, u, w, w, w]$	$\mathbf{X_c} = [6, -8, 0, 4, 0, -8]$
$\mathbf{x_4} = [u, w, u, u, w, u]$	$\mathbf{X_d} = [6, -1.07, 3.46, 4, -3.46, -14.93]$

(i) Determine which frequency domain vector corresponds to which time domain vector and give a justification for your choice in each case. [25%]

(ii) Determine the values of u and w. [5%]

(b) A real-valued zero-mean audio signal has been sampled at a sampling frequency of $f_s = 20$ kHz. A time-domain vector $\mathbf{x}$ containing 1024 consecutive samples from the audio signal has been put through the Cooley-Tukey Fast Fourier Transform (FFT) algorithm resulting in a vector $\mathbf{X}$.

(i) What is the approximate value of the first component X_0 of $\mathbf{X}$? [5%]

(ii) The 38-th value X_{37} of $\mathbf{X}$ is $4 + 3j$. Specify one other entry of $\mathbf{X}$ other than X_0 and X_{37} based on the information provided. [5%]

(iii) What real-world audio frequency does X_{37} correspond to? [10%]

(iv) Discuss the advantage of using an FFT algorithm and explain how, if at all, the outcome differs from a DFT computed from first principles. [10%]

(v) Assuming that the audio signal has a bandwidth of 4 kHz, the data can be converted to a sampling frequency $f_s' = 10$ kHz by leaving out every second sample in the time domain vector $\mathbf{x}$ and recomputing the FFT. Specify precisely how the same outcome can be obtained directly from the original FFT vector $\mathbf{X}$ without re-sampling or re-computing an FFT. [20%]

(vi) A measurement reveals that the sampling frequency of the A/D converter isn't precisely 20 kHz as advertised. A slight sampling drift results. This drift is small enough that indivdual DFT blocks can be considered unaffected, but approximately one sample is lost every 10 DFT blocks, i.e., the 10th DFT block looks as if it has a shift of one sample to the left with respect to the first block. What will be the effect of this drift in the computed DFT values? Can you suggest a way to compensate for it so that the resulting DFT values are approximately as if computed from data sampled at the correct sampling frequency? [20%]

4 The Dryden model of atmospheric turbulence is characterized by the Power Spectral Density:

$$S_X(\omega) = 2KL\frac{1 + 12L^2\omega^2}{(1 + 4L^2\omega^2)^2}$$

where K and L are scale parameters.

(a) Find the variance of this atmospheric turbulence, $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$, in terms of K and L. [30%]

Hint: use the substitution $2L\omega = \tan\theta$ as suggested in the Mathematics Data Book.

(b) Show that $S_X(\omega)$ can be written as

$$S_X(\omega) = |H(j\omega)|^2\,, \text{ where } H(j\omega) = \sqrt{2KL}\frac{1 + 2\sqrt{3}Lj\omega}{(1 + 2Lj\omega)^2}$$ [10%]

(c) Recall that if $y_h(t)$ is the response of an asymptotically linear system with frequency response $H(j\omega)$ to the input

$$u_h(t) = \sum_{k=-N/2}^{N/2-1} p(t - kh)W_k$$

where $p(t) = H(t) - H(t - h)$, $H(t)$ is the Heaviside step function and W_k are i.i.d random variables, then it is possible to scale the variance of the W_k so that

$$\lim_{h\to 0} \frac{1}{T} E[|Y_h(j\omega)|^2] = |H(j\omega)|^2$$

where Y_h is the Fourier Transform of y_h and $T = Nh$ is the length of the signal $u_h(t)$. How should the variance of the variables W_k be scaled to achieve this? [15%]

(d) For aircraft testing purposes it is required to produce a random signal with properties close to the ideal Dryden model for $K = 10$ and $L = 0.1$

(i) Calculate the step response invariant digital filter, with sampling period h, corresponding to $H(j\omega)$ in part (b). [30%]

(ii) Explain why the output of this filter in response to a realisation of the sequence $\{W_k\}$ from part (c) might be suitable for this purpose. What would be an appropiate sampling period h and why is the step response invariant transformation the appropriate transformation to be used in this case? [15%]

END OF PAPER

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