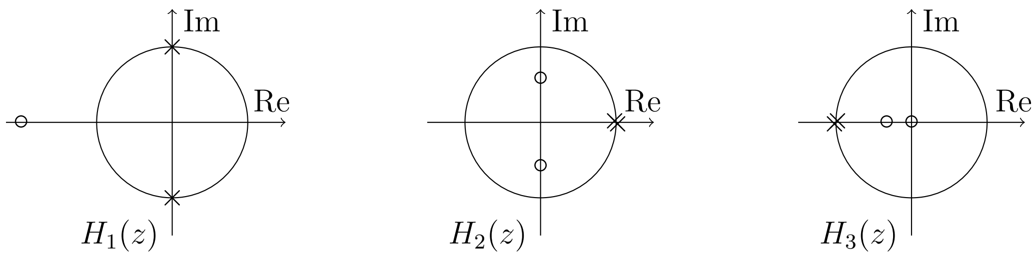
# Crib of 3F1 exam 2024

May 24, 2024

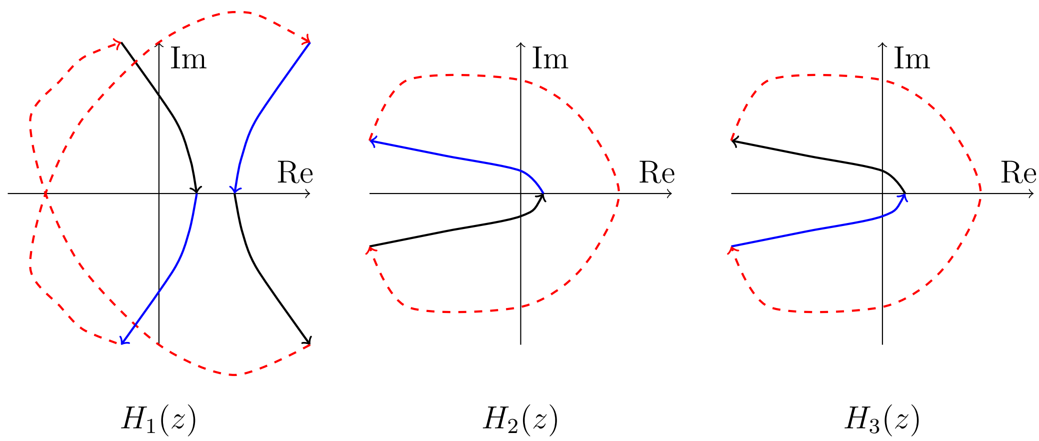
1. (a) The pole zero diagrams in the Argand plane are below.




(b) $H_1(z)$ has poles on the unit circle for $\theta = \pi/2$ and hence must be going to infinity and coming back within the range 0 to $\pi$. This is (b).

Graphs (a) and (c) are similar in that they both touch the real axis at 2/3, corresponding to $H_2(e^{j\pi}) = H_2(-1)$ and $H_3(e^{j0}) = H_3(1)$. However, as $H_3(e^{j\theta})$ leaves the real axis, its angle becomes positive because the angle from the zeros on the Argand diagram is larger than the angle from the poles. Hence (a) is $H_3(z)$ and (c) is $H_2(z)$.

(c) The complete Nyquist diagrams are given below.



(d) All 3 systems can be made stable. For $H_1(z)$, $-1/K > 3/2$ results in a stable system, hence $-2/3 < K < 0$. For both $H_2(z)$ and $H_3(z)$, $-1/K < 1/3$ results in stable systems, hence $K < -3$ or $K > 0$.

(e) The output of the system is

$$Y(z) = H_1(z)X(z) = \frac{z+2}{z^2+1} \cdot \frac{1}{1+z^{-2}} = \frac{z^3+2z^2}{(z^2+1)^2}$$

and hence using the hint we obtain for $k \geq 0$,

$$y_k = -\frac{k+1}{2} \cos \frac{\pi}{2}(k+1) - k \cos \frac{\pi}{2} k, \text{ which grows linearly with } k.$$

(f) The closed loop transfer function is

$$G(z) = \frac{KH_1(z)}{1+KH_1(z)} = \frac{-\frac{1}{2}\frac{z+2}{z^2+1}}{1-\frac{1}{2}\frac{z+2}{z^2+1}} = \frac{-1-\frac{1}{2}z}{z^2+1-1-\frac{1}{2}z} = -\frac{1}{2} \cdot \frac{z+2}{z\left(z-\frac{1}{2}\right)}$$

which is clearly a stable system as its poles are inside the unit circle. As $K = -1/2$ lies within the stability range we predicted by analyising the Nyquist diagram, this is reassuring. We obtain the stationary response by computing

$$G(e^{j\frac{\pi}{2}}) = -\frac{1}{2} \cdot \frac{j+2}{j(j-\frac{1}{2})} = \frac{1}{2} \cdot \frac{2+j}{1+\frac{1}{2}j} = 1$$

and hence the stationary response is simply equal to the input $\left\{\cos \frac{\pi}{2} k\right\}_{k \geq 0}$.

2. (a) (i) We write the equations in the $z$ domain

$$\begin{cases} U(z) + z^{-1}Y(z) = Y(z) \\ 4W(z) + 2z^{-1}Y(z) = 3X(z) \\ 2Y(z) + 2z^{-2}W(z) = 3z^{-1}U(z) \end{cases}$$

then resolve them to obtain

$$G(z) = \frac{Y(z)}{X(z)} = \frac{\frac{3}{2}z^{-2}}{z^{-3} - 3z^{-2} + 3z^{-1} - 2} = \frac{-\frac{3}{4}z}{z^3 - \frac{3}{2}z^2 + \frac{3}{2}z - \frac{1}{2}}$$

(ii) We compute $Y(z) = G(z)X(z) = G(z)\frac{z}{z+1}$ (noting that $\cos \pi k = (-1)^k$)

$$\begin{aligned} Y(z) &= \frac{-\frac{3}{4}z^2}{(z+1)(z-\frac{1}{2})(z^2-z+1)} = \frac{1/6}{z+1} - \frac{1/6}{z-\frac{1}{2}} - \frac{1/2}{z^2-z+1} \\ &= \frac{1}{6}z^{-1}\frac{1}{1+z^{-1}} - \frac{1}{6}z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{2\sin\frac{\pi}{3}}z^{-1}\frac{\sin\frac{\pi}{3}z^{-1}}{1-2\cos\frac{\pi}{3}z^{-1}+z^{-2}} \end{aligned}$$

and hence the output sequence is

$$y_k = \frac{1}{6}\cos\left(\pi(k-1)\right) - \frac{1}{6}\left(\frac{1}{2}\right)^{k-1} - \frac{\sqrt{3}}{3}\sin\left(\frac{\pi}{3}(k-1)\right)$$

(b) (i)
- $\mathbf{x_1}$ is a real vector that isn't symmetric ($x_1 \neq x_5$) so its DFT must be complex but obey conjugate symmetry: $\mathbf{X_f}$
- $\mathbf{x_2}$ is a complex vector that is not conjugate symmetric ($x_3$ is complex and hence not its own conjugate) and every even component is zero. Its DFT should be complex, not symmetric, and its second half should be equal to minus its first half: $\mathbf{X_a}$.
- $\mathbf{x_3}$ is a complex vector that obeys conjugate symmetry and its second half equals minus its first half, so its DFT should be real not symmetric, and every even entry should be zero: $\mathbf{X_e}$.
- $\mathbf{x_4}$ is a real symmetric vector whose second half equals its first half so its DFT should be real, symmetric, and every odd entry should be zero: $\mathbf{X_b}$.
- $\mathbf{x_5}$ is a complex vector that obeys conjugate symmetry and every odd entry is zero, so its DFT should be real, not symmetric, and its second half equals its first half: $\mathbf{X_c}$
- $\mathbf{x_6}$ is a real symmetric vector, so its DFT should be real, symmetric: $\mathbf{X_d}$.

(ii) The first component of $\mathbf{X_d}$ is $u - w$ should be the sum of $\mathbf{x_6}$, 2, and the first component of $\mathbf{X_b}$ is $u$ should be the sum of $\mathbf{x_4}$, 3. Hence $u = 3$ and $w = 1$.

3. (a) (i) The normalised corner frequency is $\theta_0 = 2\pi(8/48) = \pi/3$. We pre-warp this frequency into an analogue corner frequency $\omega_0 = \tan(\theta_0/2) = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ and design the digital filter by applying the bilinear transform to the filter $\frac{1}{1+s/\omega_0}$

$$H(z) = \frac{1}{1 + \frac{1}{\omega_0} \cdot \frac{z-1}{z+1}} = \frac{z+1}{z+1+\sqrt{3}(z-1)} = \frac{1}{\sqrt{3}+1} \cdot \frac{z+1}{z - \frac{\sqrt{3}-1}{\sqrt{3}+1}}$$
$$= \frac{\sqrt{3}-1}{2} \cdot \frac{z+1}{z-2+\sqrt{3}}$$

(ii) The filter has a pole at $2 - \sqrt{3} = 0.268$ and a zero at $-1$.

(iii) We compute

$$|H(e^{j\pi/3})| = \frac{\sqrt{3}-1}{2} \left| \frac{\frac{1}{2} + j\frac{\sqrt{3}}{2} + 1}{\frac{1}{2} + j\frac{\sqrt{3}}{2} - 2 + \sqrt{3}} \right| = \frac{\sqrt{3}-1}{2} \sqrt{\frac{\frac{9}{4} + \frac{3}{4}}{(\frac{3}{2} - \sqrt{3})^2 + \frac{3}{4}}}$$
$$= \sqrt{\frac{3(\sqrt{3}-1)^2}{4(6-3\sqrt{3})}} = \sqrt{\frac{12-6\sqrt{3}}{24-12\sqrt{3}}} = \frac{\sqrt{2}}{2}$$

(Note that a calculator-based solution is sufficient to get full points…)

$$20\log_{10}\frac{\sqrt{2}}{2} = -10\log_{10} 2 \approx -3\,\text{dB}.$$

(b) (i) We compute

$$h_k = \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} e^{jk\theta}d\theta = \frac{1}{2\pi kj}(e^{jk\pi/3} - e^{-jk\pi/3}) = \frac{\sin(\frac{\pi}{3}k)}{\pi k} = \frac{1}{3}\,\text{sinc}\left(\frac{\pi}{3}k\right)$$

where $\text{sinc}(x) = \frac{\sin(x)}{x}$.

(ii) We compute the five samples around zero as

$$[h_{-2}, h_{-1}, h_0, h_1, h_2] = \left[\frac{\sqrt{3}}{4\pi}, \frac{\sqrt{3}}{2\pi}, \frac{1}{3}, \frac{\sqrt{3}}{2\pi}, \frac{\sqrt{3}}{4\pi}\right] = [0.14, 0.28, 0.33, 0.28, 0.14]$$

and shift to make it causal $[h_0, h_1, h_2, h_3, h_4] = [0.14, 0.28, 0.33, 0.28, 0.14]$ and state the transfer function of the FIR

$$H(z) = 0.14 + 0.28z^{-1} + 0.33z^{-2} + 0.28z^{-3} + 0.14z^{-4}.$$

(iii) We evaluate

$$|H(e^{j\pi/3})| = \left|0.14 + 0.28e^{-j\pi/3} + 0.33e^{-j2\pi/3} + 0.28e^{-j\pi} + 0.14e^{-j4\pi/3}\right| = 0.47$$

The exact result is $|H(e^{j\pi/3})| = \frac{1}{3} + \frac{\sqrt{3}}{4\pi}$. Note that this absolute gain is misleading as the gain at $\theta = 0$ is not 1 as in the previous question. The gain relative to the gain at frequency zero can be computed as

$$\frac{|H(e^{j\pi/3})|}{|H(e^{j0})|} = \frac{\frac{1}{3} + \frac{\sqrt{3}}{4\pi}}{\frac{1}{3} + \frac{6\sqrt{3}}{4\pi}} = 0.4061$$

corresponding to a $-7.8$dB gain. Computing the absolute gain was enough to get full points on this question.

(c) (i) We obtain the step response by inverse Laplace transform of $H(s)/s$, for which we need partial fractions

$$U(s) = \frac{1}{s} - \frac{1}{s + \omega_1}$$

which gives the time domain function $u(t) = H(t) - e^{-\omega_1 t}$, where $H(t)$ is the Heaviside step function, not to be confused with the analogue filter transfer function in this question.

(ii) We sample the step response to give

$$u_k = H(kT) - e^{-\omega_1 kT}$$

where $\omega_1 T = \omega_1/f_s = \pi/3$. We then divide the digital step response in the $z$ domain by a digital step $\frac{1}{1-z^{-1}}$ to obtain

$$G(z) = (1 - z^{-1})\sum_{k=0}^{\infty}(1 - e^{k\pi/3})z^{-k} = 1 - \frac{1 - z^{-1}}{1 - e^{\pi/3}z^{-1}} = \frac{1 - e^{\pi/3}}{z - e^{\pi/3}}$$

Its gain for $z = e^{j\pi/3}$ is

$$|G(e^{j\pi/3})| = \left| \frac{1 - e^{\pi/3}}{e^{j\pi/3} - e^{\pi/3}} \right| = 0.74$$

and $20 \log_{10} |G(e^{j\pi/3})| = -2.63\,\text{dB}$.

4. (a) The PSD is the Fourier transform of the auto-correlation function, i.e.,

$$\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{\infty} r_{XX}(t) e^{-j\omega t} dt \\ &= \frac{\tau}{2} \left( \int_{-\infty}^{0} e^{(\frac{1}{\tau} - j\omega)t} dt + \int_{0}^{\infty} e^{-(\frac{1}{\tau} + j\omega)t} dt \right) \\ &= \frac{\tau^2}{2} \left( \frac{1}{1 - j\omega\tau} + \frac{1}{1 + j\omega\tau} \right) = \frac{\tau^2}{1 + \tau^2\omega^2} \end{aligned}$$

(b) The output $Y$ of the linear system has PSD

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$H(\omega)$ is the Fourier transform of the impulse response $h(t) = e^{-\alpha t}$,

$$H(\omega) = \int_0^{\infty} e^{-\alpha t} e^{-j\omega t} dt = \frac{1}{\alpha + j\omega}$$

hence

$$\begin{aligned} S_{YY}(\omega) &= \frac{1}{\alpha^2 + \omega^2} \cdot \frac{\tau^2}{1 + \tau^2\omega^2} \\ &= \frac{\tau^2}{1 - \tau^2\alpha^2} \left( \frac{1/\alpha^2}{1 + \omega^2/\alpha^2} - \frac{\tau^2}{1 + \omega^2\tau^2} \right) \end{aligned}$$

By inspection from (a) this gives an auto-correlation function for $Y$ of

$$R_{YY}(t) = \frac{\tau^2}{1 - \tau^2\alpha^2} \left( \frac{1}{2\alpha} e^{-\alpha|t|} - \frac{\tau}{2} e^{-|t|/\tau} \right)$$

and

$$\sigma_Y^2 = R_{YY}(0) = \frac{\tau^2}{1 - \tau^2\alpha^2} \left( \frac{1}{2\alpha} - \frac{\tau}{2} \right) = \frac{\tau^2/\alpha^2}{2\,(\tau + 1/\alpha)}$$

(c) The process $X$ can be obtained by filtering white noise through a filter $G(\omega) = \frac{\tau}{1 + j\omega\tau}$, which relates an input signal $U(\omega)$ to an output signal $X(\omega)$ via the equation $X(\omega) = G(\omega)U(\omega)$ and hence $X(\omega) + j\omega\tau X(\omega) = \tau U(\omega)$, corresponding to the differential equation

$$x + \tau \frac{dx}{dt} = \tau u.$$

The discrete system proposed in the question is an Euler approximation of this differential equation, i.e.,

$$x(k\delta) + \tau \frac{x((k+1)\delta) - x(k\delta)}{\delta} = \tau u_k.$$

Re-arranging, we obtain

$$x((k+1)\delta) = \left(1 - \frac{\delta}{\tau}\right) x(k\delta) + \delta u_k$$

implying that $a = \delta/\tau$. As we learned in lectures, to obtain a discrete approximation of a continuous random process that converges to the correct process as $\delta$ tends to zero, the input zero-mean i.i.d. random variables need to be normalised so their variance scales with $1/\delta$. Hence, assuming that $u_k = cw_k$, i.e., the variables $u_k$ in our equations are the scaled unit variance variables $w_k$ in the question, we conclude that $b = \sqrt{1/\delta}$.