Version JS/2 with answers

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday, 23 April 2024 9.30 to 11.10

Module 3F1

SIGNALS & SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) For each of the three systems with transfer functions

$$H_1(z) = \frac{z+2}{z^2+1}$$
$$H_2(z) = \frac{z^2+\frac{1}{3}}{(z-1)^2}$$
$$H_3(z) = \frac{z\left(z+\frac{1}{3}\right)}{(z+1)^2}$$

draw a pole-zero diagram in the Argand plane. [10%]

(b) Figure 2 contains the Nyquist diagrams of the three systems defined in part (a) drawn as the normalised frequency θ goes from 0 to π in shuffled order. Assign the correct Nyquist diagram to each of those three systems and justify your choice. [20%]

(c) Sketch the complete Nyquist diagrams for the three systems. [30%]

(d) For each of the three systems, determine whether the system can be stabilised using a proportional controller K in a closed loop system as indicated in Fig. 1 and if so, specify a stability range for the controller K. [10%]

(e) Determine the output sequence y_k of the system with transfer function $H_1(z)$ for the bounded input sequence $\{x_k\}_{k\geq 0} = \left\{\cos\frac{\pi}{2}k\right\}_{k\geq 0}$ and verify that it is unbounded. [15%]
Hint: the z transform of $\left\{\frac{k}{2}\cos\frac{\pi}{2}k\right\}_{k\geq 0}$ is $\frac{-z^2}{(z^2+1)^2}$.

(f) Now consider the system with transfer function $H_1(z)$ placed in a closed loop control system as indicated in Fig. 1 with $K = -1/2$. Show that the closed loop system is stable and hence determine the stationary response to the input sequence $\{x_k\}_{k\geq 0} = \left\{\cos\frac{\pi}{2}k\right\}_{k\geq 0}$ [15%]

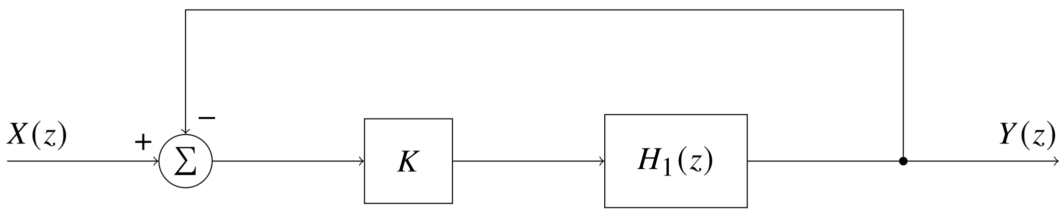

Fig. 1

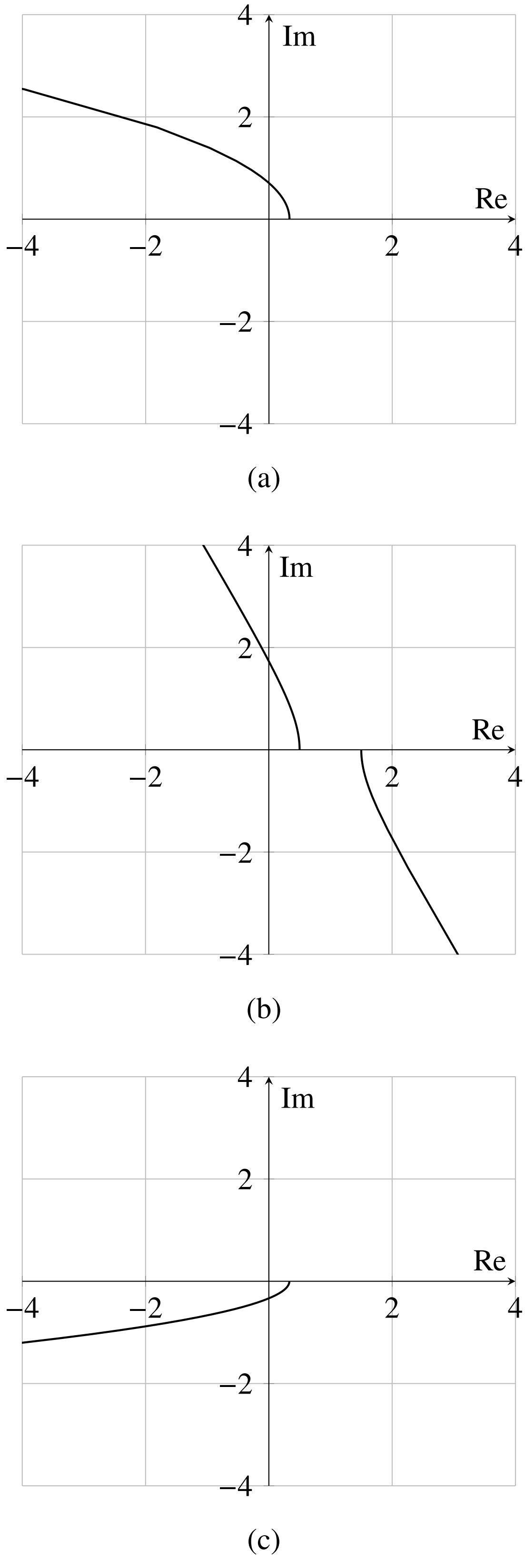

Fig. 2

2 (a) Consider the system of difference equations

$$\begin{cases} u_k + y_{k-1} = y_k \\ 4w_k + 2y_{k-1} = 3x_k \\ 2y_k + 2w_{k-2} = 3u_{k-1} \end{cases}$$

where all signals are zero for $k < 0$.

(i) Compute the transfer function $G(z)$ of the system that maps the input $X(z)$ to the output $Y(z)$. [10%]
Hint: $G(z)$ has a pole at $z = 1/2$.

(ii) For $x_k = \cos(k\pi)$ for $k \geq 0$, determine the output sequence $\{y_k\}_{k\geq 0}$. [40%]

(b) Consider the following two lists of time domain vectors and of Discrete Fourier Transform (DFT) frequency domain vectors. u and w are non-zero real numbers.

Time domain	Frequency Domain
$\mathbf{x_1} = [2, \frac{2+\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}, -\frac{1}{3}, \frac{2\sqrt{3}}{3}, \frac{2-\sqrt{3}}{3}]$	$\mathbf{X_a} = [u + jw, u - jw, u + jw, -u - jw, -u + jw, -u - jw]$
$\mathbf{x_2} = [0, 1 + j\sqrt{3}, 0, 1 + j, 0, 1 - j\sqrt{3}]$	$\mathbf{X_b} = [u, 0, w, 0, w, 0]$
$\mathbf{x_3} = [\frac{3}{2}, j\frac{\sqrt{3}}{6}, j\frac{\sqrt{3}}{6}, -\frac{3}{2}, -j\frac{\sqrt{3}}{6}, -j\frac{\sqrt{3}}{6}]$	$\mathbf{X_c} = [u + w, u - w, u + w, u + w, u - w, u + w]$
$\mathbf{x_4} = [\frac{5}{6}, \frac{1}{3}, \frac{1}{3}, \frac{5}{6}, \frac{1}{3}, \frac{1}{3}]$	$\mathbf{X_d} = [u - w, u, w, u + w, w, u]$
$\mathbf{x_5} = [\frac{10}{3}, 0, \frac{1-j\sqrt{3}}{3}, 0, \frac{1+j\sqrt{3}}{3}, 0]$	$\mathbf{X_e} = [0, u + w, 0, u, 0, u - w]$
$\mathbf{x_6} = [\frac{7}{3}, 0, \frac{1}{3}, -1, \frac{1}{3}, 0]$	$\mathbf{X_f} = [u, u + jw, w - ju, w, w + ju, u - jw]$

(i) Determine which frequency domain vector corresponds to which time domain vector and give a justification for your choice in each case. [40%]

(ii) Determine the values of u and w. [10%]

3 For an audio system with sampling rate $f_s = 48\,\text{kHz}$, a digital lowpass filter is required with corner frequency 8 kHz. In this question, we will explore different techniques for obtaining such a digital filter.

(a) A first-order analogue lowpass filter with the transfer function $H(s) = \frac{1}{1+s/\omega_0}$ has a 3dB corner frequency of ω_0 rad s^{-1}.

(i) Design the required digital filter starting from an appropriately designed first-order analogue filter using the bilinear transform $s \rightarrow \frac{z-1}{z+1}$. [20%]

(ii) Calculate the poles and zeros of the digital filter. [10%]

(iii) Show that it achieves 3dB at the required corner frequency. [10%]

(b) We now explore direct design of a Finite Impulse Response (FIR) filter in the discrete-time domain.

(i) Derive an expression for the non-causal pulse response of the ideal lowpass filter using the inverse Discrete-Time Fourier Transform (DTFT) expression $h_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})e^{jk\theta}d\theta$ applied to the desired spectrum $H(e^{j\theta}) = 1$ for θ in the required passband of the lowpass filter and $H(e^{j\theta}) = 0$ otherwise. [20%]

(ii) Truncate the response to 5 samples using a rectangular window, shift appropriately to make it causal and state the z domain transfer function of the resulting FIR filter. [10%]

(iii) Compute the filter gain at the required corner frequency. [10%]

(c) The last method applies the step invariant transform to the analogue filter $H(s) = \frac{1}{1+s/\omega_1}$ with $\omega_1 = 2\pi \times 8 \times 10^3$ Hz.

(i) Determine the step response of the analogue filter. [10%]

(ii) Obtain a discrete-time filter using the step-invariant transform, state its z domain transfer function, and compute its gain at the corner frequency. [10%]

4 A linear continuous-time system has impulse response $h(t) = e^{-\alpha t}$, $t \geq 0$. The input to this linear system is a Wide Sense Stationary process $\{x(t)\}$ with zero mean and autocorrelation function $r_{XX}(t) = \frac{\tau}{2} e^{-|t|/\tau}$.

(a) Using the fact that

$$r_{XX}(t) = \begin{cases} \frac{\tau}{2} e^{-t/\tau} & t \geq 0 \\ \frac{\tau}{2} e^{t/\tau} & t < 0 \end{cases}$$

or otherwise, show that the Power Spectral Density of X is given by

$$S_{XX}(\omega) = \frac{\tau^2}{1 + \omega^2 \tau^2}$$

[40%]

(b) Calculate the Power Spectral Density and average power of the output of the linear system. [40%]

(c) Approximate sample paths of the process X are to be generated for use as test inputs to the linear system, using a finite difference approach:

$$x((k+1)\delta) = (1 - a)x(k\delta) + bw_k$$

where the w_k are IID random variables with zero mean and unit variance. How should a and b be chosen? [20%]

END OF PAPER

Numerical Answers

1. (b) (a) is $H_3(z)$, (b) is $H_1(z)$ and (c) is $H_2(z)$.

(d) $H_1(z)$ stable for $-2/3 < K < 0$, $H_2(z)$ and $H_3(z)$ stable for $K < -3$ or $K > 0$.

(e) $y_k = -\frac{k+1}{2} \cos \frac{\pi}{2}(k+1) - k \cos \frac{\pi}{2} k$.

(f) Stable as poles at 0 and $1/2$ inside the unit circle, stationary response is $\left\{\cos \frac{\pi}{2} k\right\}_{k \geq 0}$.

2. (a) (i) $G(z) = \frac{-\frac{3}{4}z}{z^3 - \frac{3}{2}z^2 + \frac{3}{2}z - \frac{1}{2}}$

(ii) $y_k = \frac{1}{6} \cos\left(\pi(k-1)\right) - \frac{1}{6}\left(\frac{1}{2}\right)^{k-1} - \frac{\sqrt{3}}{3} \sin\left(\frac{\pi}{3}(k-1)\right)$

(b) (i) $\mathbf{x_1} \to \mathbf{X_f}, \mathbf{x_2} \to \mathbf{X_a}, \mathbf{x_3} \to \mathbf{X_e}, \mathbf{x_4} \to \mathbf{X_b}, \mathbf{x_5} \to \mathbf{X_c}, \mathbf{x_6} \to \mathbf{X_d}$

(ii) $u = 3, w = 1$.

3. (a) (i) $H(z) = \frac{\sqrt{3}-1}{2} \cdot \frac{z+1}{z-2+\sqrt{3}}$.

(ii) Pole at $2 - \sqrt{3}$ and zero at -1.

(b) (i) $h_k = \frac{1}{3} \operatorname{sinc}\left(\frac{\pi}{3} k\right)$.

(ii) $H(z) = 0.14 + 0.28z^{-1} + 0.33z^{-2} + 0.28z^{-3} + 0.14z^{-4}$.

(iii) Absolute gain 0.47 (or relative gain 0.41 or -7.8dB)

(c) (i) $u(t) = H(t) - e^{-\omega_1 t}$ where $H(t)$ is the Heaviside step function.

(ii) Gain 0.74 or -2.63dB.

4. (b) $S_{YY}(\omega) = \frac{1}{\alpha^2+\omega^2} \cdot \frac{\tau^2}{1+\tau^2\omega^2}$, $\sigma_Y^2 = \frac{\tau^2/\alpha^2}{2(\tau+1/\alpha)}$.

(c) $a = \delta/\tau, b = \sqrt{1/\delta}$.