Crib of 3F1 exam 2025

1. (a) We write the differential equations in the Laplace domain including $T_0 = \ddot{x} - a\dot{x}$,

$$\bar{T}(s) = s^2\bar{x}(s) - as\bar{x}(s) + f_0 s\bar{y}(s)\cos(\phi) + \frac{f_0}{\omega}s^2\bar{y}(s)\sin(\phi)$$
$$\bar{T}(s) = ms^2\bar{y}(s) + cs\bar{y}(s) + k\bar{y}(s)$$

then combining the two equations to eliminate $\bar{T}(s)$,

$$G(s) = \frac{\bar{y}(s)}{\bar{x}(s)} = \frac{s^2 - as}{\left(m - \frac{f_0}{\omega}\sin(\phi)\right)s^2 + (c - f_0\cos(\phi))s + k}$$

Damping is negative for $c - f_0\cos(\phi) < 0$ and $m - \frac{f_0}{\omega}\sin(\phi) > 0$, which could happen for smaller values of ϕ depending on the values of the constants.

(b) We insert the values of the constants

$$G(s) = \frac{s^2 - \frac{\log 3}{2}s}{s^2 - (\log 3)s + \left(\frac{\log 3}{2}\right)^2 + \left(\frac{2\pi}{3}\right)^2} = \frac{s^2 - \frac{\log 3}{2}s}{\left(s^2 - \frac{\log 3}{2}\right)^2 + \left(\frac{2\pi}{3}\right)^2}$$

For a D/A converter that holds the value of a signal constant between samples, a step-invariant transform is appropriate because a discrete-time step signal will map to a continuous-time step signal. Hence, we compute the step response of the system first in the Laplace domain

$$U(s) = \frac{G(s)}{s} = \frac{s - \frac{\log 3}{2}}{\left(s^2 - \frac{\log 3}{2}\right)^2 + \left(\frac{2\pi}{3}\right)^2}$$

then in the time domain

$$u(t) = e^{\frac{\log 3}{2}t}\cos\left(\frac{2\pi}{3}t\right),$$

then sample to obtain the disrete-time step response with $T = 1$,

$$u_k = e^{\frac{\log 3}{2}kT}\cos\left(\frac{2\pi}{3}kT\right) = (\sqrt{3})^k\cos\left(\frac{2\pi}{3}k\right)$$

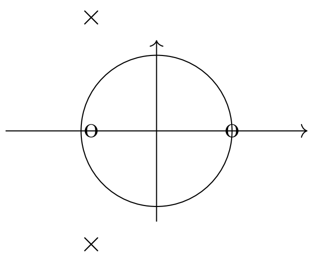
and transform to the z domain

$$U(z) = \frac{1 - \sqrt{3}z^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)}{1 - 2\sqrt{3}\left(\cos\left(\frac{2\pi}{3}\right)\right)z^{-1} + 3z^{-2}} = \frac{1 + \frac{\sqrt{3}}{2}z^{-1}}{1 + \sqrt{3}z^{-1} + 3z^{-2}}$$

and now divide by the discrete step $H(z) = \frac{1}{1-z^{-1}}$ to yield the transfer function

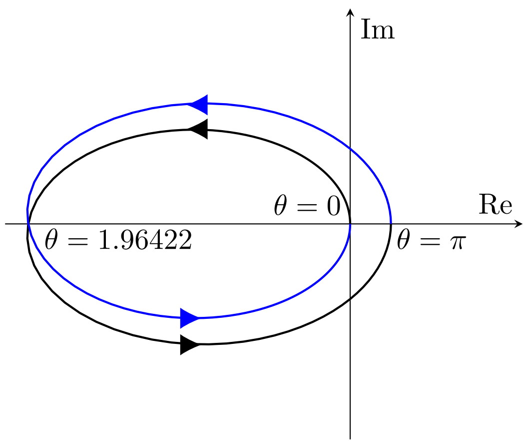
$$G(z) = (1-z^{-1})U(z) = \frac{(1-z^{-1})\left(1+\frac{\sqrt{3}}{2}z^{-1}\right)}{1+\sqrt{3}z^{-1}+3z^{-2}} = \frac{z^2-\left(1-\frac{\sqrt{3}}{2}\right)z-\frac{\sqrt{3}}{2}}{z^2+\sqrt{3}z+3}$$

(c) We solve the quadratic in the denominator to find the poles $p = -\frac{\sqrt{3}}{2} \pm j\frac{3}{2}$, and the zeros are $z_1 = 1$ and $z_2 = -\frac{\sqrt{3}}{2}$ as evident from the expression above when we multiplied $U(z)$ by $1 - z^{-1}$.



We note that there are 2 poles outside the unit circle so the system is unstable, which was to be expected since it models a continuous time system with negative damping coefficient.

(d) (i) We draw the diagram for θ from 0 to 2π:



(ii) The open loop system has two poles strictly outside the unit circle, so the closed loop system will need two encirclements of the pont $-1/K$ to achieve stability. This is the case for $-1/K$ between $H(e^{1.96422j}) = -0.933$ and $H(e^{0j}) = H(1) = 0$,

$$\begin{aligned} -0.933 < -1/K < 0 \\ 0.933 > 1/K > 0 \\ 1.072 < K < +\infty \end{aligned}$$

so we need $K > 1.072$ to achieve stability.

2. (a) The term $\frac{1}{2}(1/2)^{k-1}$ corresponds to an expression $\frac{1}{2}z^{-1}\frac{z}{z-1/2}$ in the z domain, but this expression will also give an "unwanted" contribution of $1/2$ for $k=1$. Hence, the z domain transfer function is

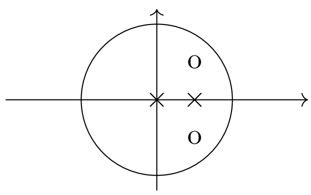
$$G(z) = 1 - z^{-1} + \frac{1/2}{z-1/2}$$

where the term $-z^{-1}$ corrects the first term $1/2$ of the geometric sequence to $-1/2$.

(b) We manipulate $G(z)$ to obain a rational expression

$$\begin{aligned} G(z) &= \frac{z-1}{z} + \frac{1/2}{z-1/2} = \frac{(z-1)(z-1/2)+z/2}{z(z-1/2)} \\ &= \frac{z^2-z+1/2}{z(z-1/2)} \end{aligned}$$

The zeros are $z = \frac{1}{2} \pm \frac{1}{2}j$, hence



(c) We re-write the transfer function in terms of z^{-1}

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}+z^{-2}/2}{1-z^{-1}/2}$$

and expand as

$$Y(z) - z^{-1}Y(z)/2 = X(z) - z^{-1}X(z) + z^{-2}X(z)/2$$

to obtain the difference equation

$$y_k - \frac{1}{2}y_{k-1} = x_k - x_{k-1} + \frac{1}{2}x_{k-2}$$

(d) We compute the sum of the absolute values of g_k

$$\sum_{k=0}^{\infty}|g_k| = 1 + \left|-\frac{1}{2}\right| + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2$$

Since the sum is finite, the system is stable. This can also be verified by the fact that the poles $z=0$ and $z=1/2$ lie inside the unit circle, but the question specifically asks to verify stability via the delta response.

(e) We can use the final value theorem, since the system is stable

$$\lim_{k\to\infty} w_k = \lim_{z\to\infty}(z-1)W(z) = \lim_{z\to\infty}(z-1)\frac{1}{z-1}G(z) = G(1) = 1$$

(f) We evaluate the tranfer function at the normalised frequency $\theta = \pi/2$

$$G(e^{j\pi/2}) = G(j) = \frac{-1/2 - j}{-1 - j/2} = \frac{(-1/2 - j)(-1 + j/2)}{(-1)^2 - (j/2)^2} = \frac{4}{5} + \frac{3}{5}j = e^{j\tan^{-1}(3/4)}$$

Hence the output signal b_k is

$$b_k = \sin\left(\frac{\pi}{2}k + \tan^{-1}(3/4)\right) = \sin\left(\frac{\pi}{2}k + 0.64\right)$$

3. (a) First, we note that $\mathbf{x}$ is a real vector that obeys the symmetry rules for the DFT ($x_k = x^*_{N-k}$) so its DFT must be real (due to the symmetry of $\mathbf{x}$) and symmetric (because $\mathbf{x}$ is real). In other words the DFT $\mathbf{X}$ of $\mathbf{x}$ is also real and symmetric.

1. Filter 1 is a real filter, so $\mathbf{Y}$ remains real and hence $\mathbf{y}$ must obey conjugate symmetry. The filter won't maintain symmetry so we expect $\mathbf{y}$ to be complex. Furthermore, every odd component of $\mathbf{Y}$ is zeroed by the filter, so we expect the second half of $\mathbf{y}$ to repeat the first half, i.e., $y_{k+N/2} = y_k$ for $N = 8$. This combination of properties is only satisfied by $\mathbf{y_c}$.
2. Filter 2 is a complex filter, so $\mathbf{Y}$ is complex and hence $\mathbf{y}$ does not satisfy the symmetry property. The filter also doesn't maintain the symmetry propery in the frequency domain, so $\mathbf{y}$ must be complex. Of the 3 remaining complex options, only $\mathbf{y_e}$ doesn't fulfil the symmetry property.
3. Filter 3 is real so $\mathbf{Y}$ is real and $\mathbf{y}$ must be symmetric. The filter is also symmetric so will maintain the symmetry property in the frequency domain, hence $\mathbf{y}$ mst be real. Of the two real options, only $\mathbf{y_f}$ is symmetric.
4. Filter 4 is complex and symmetric, hence $\mathbf{y}$ must be real but not symmetric. That's option $\mathbf{y_a}$.
5. Filter 5 is real but not symmetric, hence $\mathbf{y}$ must be symmetric and complex. The fact that the filter fulfils the repetition property $F_{k+N/2} = F_k$ is irrelevant because the signal $\mathbf{X}$ doesn't fulfil this property, so neither will $\mathbf{Y}$, so we don't expect to find zeros in even positions of $\mathbf{y}$ (indeed, there are no solutions that have this property.) At this point, the properties we identified cannot decide between the two remaining complex symmetric options, so we skip to Filter 6.
6. Filter 6 is real, not symmetric, and zeros every even component of $\mathbf{Y}$, so y must be symmetric, complex, and fulfil the negative repetition property $y_{k+N/2} = -y_k$. Only $\mathbf{y_b}$ fulfils these 3 properties.

By exclusion, Filter 5 corresponds to $\mathbf{y_d}$.

(b) (i) Since the signal is real, $X(5\pi/3) = X(-\pi/3) = X^*(\pi/3) = 1.2 - j$.

(ii) The signal needs to satisfy the conjugate symmetry property $x_{-k} = x^*_k$. Since the signal is real, this reduces to the condition $x_{-k} = x_k$ that the signal is even.

(iii) Using the time delay operator:

$$Y(\theta) = e^{-jd\theta} X(\theta)$$

Unlike the z transform there is no need to adjust for values at negative times, since the DTFT is a two-sided transform.

(iv) The convolution property gives

$$Y(\theta) = G(\theta) X(\theta)$$

(v) This gives the periodic convolution in the frequency domain,

$$Y(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\varphi) X(\theta - \varphi) d\varphi$$

4. (a) We transform the differential equation into the Fourier domain

$$Y(\omega) = j\omega \frac{L}{R} jY(\omega) = X(\omega)$$

and obtain the circuit transfer function

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 + j\omega \frac{L}{R}}.$$

The power spectral density of Y is obtained by multiplying with the squared magnitude of the filter

$$\mathcal{S}_{YY}(\omega) = |G(j\omega)|^2 \mathcal{S}_{XX}(\omega) = \frac{N_0}{1 + \omega^2 \left(\frac{L}{R}\right)^2} = \frac{N_0 R}{2L} \frac{2R/L}{\left(\frac{R}{L}\right)^2 + \omega^2}.$$

Using the Fourier transform expression given, we conclude that

$$R_{YY}(\tau) = \frac{N_0 R}{2L} e^{-R|\tau|/L}$$

The mean power is $E[Y^2(t)] = R_{YY}(0) = \frac{N_0 R}{2L}$.

(b) White noise was illustrated in lectures as the limit of a staircase process when the width of the rectangular "stairs" goes to zero and the variance of the random variables multiplying them grows inversely proportional to the width. Hence, in its limit, white noise has infinite power and is discontinuous at every point, both of which are not possible in the real world. We can only ever encounter the equivalent of filtered white noise in nature.

(c) The power spectral density of the bandlimited input noise is N_0 for ω between $-2\pi f_0$ and $+2\pi f_0$ and zero elsewhere, and hence the input power is, by Parseval,

$$E[X^2] = \frac{1}{2\pi} \int_{-2\pi f_0}^{2\pi f_0} N_0 d\omega = 2N_0 f_0$$

$\mathcal{S}_{YY}(\omega)$ is the same as previously, except that it is zero outside the range between $-2\pi f_0$ and $+2\pi f_0$, so to obtain the output power $E[Y^2]$, we use Parseval again

$$E[Y^2] = \frac{1}{2\pi} \int_{-2\pi f_0}^{2\pi f_0} \mathcal{S}_{yy}(\omega) d\omega = \frac{N_0 R^2}{2\pi L^2} \int_{-2\pi f_0}^{2\pi f_0} \frac{1}{\omega^2 + \left(\frac{R}{L}\right)^2} d\omega$$

As suggested, we substitute $\omega = \frac{R}{L} \tan\theta$, noting that

$$d\omega = \frac{R}{L} \frac{1}{\cos^2\theta} d\theta$$

and $\theta = \tan^{-1}(L\omega/R)$ so that the integral boudaries become $\pm\theta_0$ where $\theta_0 = \tan^{-1}(2\pi f_0 L/R)$, yielding

$$E[Y^2] = \frac{N_0 R^2}{2\pi L^2} \int_{-\theta_0}^{\theta_0} \frac{R/L}{\left(\frac{R}{L}\tan\theta\right)^2 + \left(\frac{R}{L}\right)^2} \frac{1}{\cos^2\theta} d\theta = \frac{N_0 R}{2\pi L} \int_{-\theta_0}^{\theta_0} \frac{1}{\sin^2\theta + \cos^2\theta} d\theta$$

$$= \frac{N_0 R}{2\pi L} \int_{-\theta_0}^{\theta_0} d\theta = \frac{2N_0 R\theta_0}{2\pi L} = \frac{2N_0 R}{2\pi L} \tan^{-1}\left(\frac{2\pi L f_0}{R}\right)$$

Hence the ratio of output to input power is $\frac{R}{2\pi f_0 L} \tan^{-1}\left(\frac{2\pi f_0 L}{R}\right)$.

Question 1: Nyquist Diagrams and Stability
Attempts: 145, mean 10.9/20, highest 20/20, lowest 3/20
A popular and easy question that was generally done well. 11 students got 20/20 on this question. The average mark on this question is weighed down by a small group of students who barely attempted the question, but if you exclude those the average on the question would be quite high. There was an error in the question: the Nyquist diagram was drawn for normalised frequencies θ from π to 2π instead of 0 to π as stated. The complete Nyquist diagram obtained from this graph has the correct shape but the opposite direction (clockwise instead of anti-clockwise). Only 19 students noticed this and concluded that the system with 2 open-loop poles cannot be stabilised in a feedback loop for any linear gain *K*, which is the correct conclusion given the wrong graph they were given. All of those 19 students received the full 8 points for this sub-question. The list of candidate numbers of those 19 students is given as an appendix to this report. The remaining 126 students who took this question ignored the direction of the graph and those who were able to derive the inequalities correctly found the correct stability range. Those who did also got the full 8 points for this sub-question even though one might have considered penalising them for forgetting to check the graph direction, but on the other hand it wouldn't have been fair to delete points for obtaining what is, after all, the correct stability range for the system. After examining all the papers, it is also clear that this mistake did not cause a loss of time or extra work to any student. The lesson learned for me is to be more careful in drawing Nyquist diagrams and also to insist more on the importance of the direction of the curve during lectures as it's disappointing that only 13% of students remembered that the graph direction matters. An updated question has been produced and will be uploaded to the department server so that future generations of students aren't confused by this when revising.

Question 2: z Transform, delta response, difference equation and transfer function
Attempts: 164, mean 9/20, highest 20/20, lowest 1/20
The most popular question, answered well by many students. Some students stumbled on the first step, to obtain the z transform of a delta response, which required some fiddling since this was not simply a geometric sequence and required a correction term, but the vast majority got this right. Most of those who did also found it easy to work back to the difference equation. The stability question was also mostly answered well, although some thought they needed to take the sum of the delta response rather than the sum of absolute values, and some forgot completely how stability could be verified from the delta response and argued stability based on the pole positions (which was not what had been asked so did not give points. In part (e) students were asked to apply the final value theorem and it was important to mention that this is only possible because the system is stable. Finally, the last question was generally well answered but a few students didn't understand what had been asked and tried to determine the full response to the input signal including the transient and stationary part, despite the fact that the question used both terminologies that had been used in the course and in past exams ("stationary" and "steady-state") to make it clear that the transient was not required.

Question 3: DFT / DTFT
Attempts: 117, mean 11.8/20, highest 20/20, lowest 2/20
The first half of this question tested students' understanding of the properties of the DFT by asking them to assign frequency-domain filter masks to their resulting time-domain results when filtering a vector ***x***. A similar format of question has been asked in two pasts exams and in a new examples paper question so students had had ample opportunity to practice this format. The majority of students missed the added subtlety that the results were not merely the inverse DFT of the filter masks but depended on the properties of the filtered vector ***x***. Those who obtained the correct assignment but not commenting on the properties of ***x*** (conjugate symmetric and real) that justifies this assignment lost 2 out of 10 points. The second half of the question asked about properties of the DTFT, which had been emphasised in lectures this year more than in previous years, and whose properties were fully covered in the course. Many students found this easy and clocked easy points on this part of the question, while some got predictably confused between the DTFT, the DFT and the *z* transform and missed out on those easy points. The last question that asked to state that multiplication of discrete time signals in the time domain gives the periodic (circular) convolution in the DTFT domain had fewer correct answers as many students thought this would be a linear convolution, forgetting that the spectrum of a discrete-time signal is always periodic.

Question 4: Random Processes
Attempts: 93, mean 8.9/20, highest 20/20, lowest 1/20
The least popular question but it was generally done very well and many students found it easy. The average on this question is misleading because it is weighed down by some students who barely attempted the question, while those who did it seriously got good marks. The question was a direct application of what was learned in lectures, and the conceptual part (b) of the question had been asked almost identically in a previous exam. Most answered this question perfectly of course, but some who didn't get it provided a surprisingly wide variety of somewhat strange justifications, going from "no real signal has zero mean" to "signals cannot truly be Gaussian" (in fact a zero mean white process does not need to be Gaussian anyway and the question did not state this, quite apart from the fact that there are many quantities in the real world that are very well modelled by a Gaussian random variable). We were looking for an answer that mentioned infinite power and possibly discontinuity at every instant in time. Some students wrote "infinite energy" instead of power and this was accepted although technically irrelevant (e.g. an infinite length sinusoid has infinite energy but that doesn't make it unrealistic.) In future, we will emphasise the difference between power and energy in lectures.